1 Conjugation from group translation

Group action and group-set: A (left) action of a group G on a set X is a map

$$A\colon G \times X \to X, \quad (g,x) \mapsto g \cdot x = gx \tag{1}$$

such that $\forall g_1, g_2 \in G$ and $x \in X$:

- $g_1(g_2 x) = (g_1 g_2)x$
- $e_G x = x$

or, equivalently, the map $G \to \mathrm{Sym}(X)$ given by $g \mapsto g \cdot (-)$ is a homomorphism. The pair (X, A) is called a G-set, and A is called a G-action.

Group translation is action: G-action on underlying space of G: $gx := L_g(x)$. Denote this group-set as (G, L). NB: the object (G, L) is a group-set, not a group!

Map of group-sets: Let X_i be a G_i-set for $i = 1, 2$. A function $f\colon X_1 \to X_2$ is a map of group-sets if it is equivariant, i.e. $\exists$ homomorphism $\xi\colon G_1 \to G_2$ such that

$$f(hx) = \xi(h) f(x), \quad \forall h \in G_1, x \in X_1. \tag{2}$$

Conjugation from translation: For each $g \in G$, the map L_g is a map from the group-set (G, L) to itself, equivariant w.r.t. the homomorphism $C_g\colon G \to G$, $h \mapsto ghg^{-1}$.

Proof.

$$L_g(hx) = ghx = ghg^{-1}gx = (ghg^{-1})(gx) = C_g(h) L_g(x). \tag{3}$$

□

2 Conjugation and automorphisms

Automorphism: An automorphism of G is a group isomorphism $\phi\colon G \to G$, the set of all automorphisms of G is the automorphism group $\mathrm{Aut}(G)$.

Group center, inner & outer automorphisms: The map $C\colon G \to \mathrm{Aut}(G)$, $g \mapsto C_g$ is a homomorphism. From this map C follows:

- the center $Z(G)$ as $\ker(C)$
- the group $\mathrm{Inn}(G)$ of inner automorphisms as $\mathrm{im}(C)$
- the group $\mathrm{Out}(G) := \mathrm{Aut}(G)/\mathrm{Inn}(G)$ of outer automorphisms as $\mathrm{coker}(C)$

Summarized in exact sequence (sequence where image of one map is kernel of next map)

$$1 \longrightarrow Z(G) \xrightarrow{\text{inclusion}} G \xrightarrow{C} \mathrm{Aut}(G) \xrightarrow{\text{quotient}} \mathrm{Out}(G) \longrightarrow 1 \tag{4}$$

3 Derivatives of conjugation

Assume G is Lie group with algebra $\mathfrak{g} = T_e G$.

Adjoint representation: Differentiate conjugation at identity:

$$\mathrm{Ad}_g = T_e C_g\colon \mathfrak{g} \to \mathfrak{g}, \quad X \mapsto \left.\frac{d}{dt}\right|_{t=0} C_g(\exp(tX)) = \text{"}gXg^{-1}\text{"} \tag{5}$$

yields a smooth homomorphism

$$\mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g}), \quad g \mapsto \mathrm{Ad}_g\,. \tag{6}$$

Little adjoint/bracket: Differentiating this map at identity yields

$$\mathrm{ad} = T_e\,\mathrm{Ad}\colon \mathfrak{g} \to \mathrm{End}(\mathfrak{g}), \quad X \mapsto \mathrm{ad}(X) := \left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}_{\exp(tX)}\,. \tag{7}$$

and it can be shown that

$$\mathrm{ad}(X)Y = [X, Y]. \tag{8}$$

Bracket as second derivative: Use group commutator $[g_1, g_2] = g_1 g_2 g_1^{-1} g_2^{-1}$. One has

$$[X, Y] = \frac{d}{dt}\Big|_{t=0} \frac{d}{ds}\Big|_{s=0} [\exp(tX), \exp(sY)] = \frac{d}{dt}\Big|_{t=0} \frac{d}{ds}\Big|_{s=0} \exp(tX)\exp(sY)\exp(-tX)\exp(-sY) \tag{9}$$

note that the identity $[g_1, g_2] = [g_2, g_1]^{-1}$ implies $[X, Y] = -[Y, X]$.

Jacobi identity: As Ad is map of groups and $\mathrm{ad} = T_e \,\mathrm{Ad}$, it follows that ad is map of algebras. Hence for $X, Y \in \mathfrak{g}$;

$$\mathrm{ad}([X, Y]) = [\mathrm{ad}(X), \mathrm{ad}(Y)] = \mathrm{ad}(X)\,\mathrm{ad}(Y) - \mathrm{ad}(Y)\,\mathrm{ad}(X). \tag{10}$$

Applying this map to $Z \in \mathfrak{g}$ yields Jacobi identity;

$$[[X, Y], Z] = [X, [Y, Z]] - [Y, [X, Z]]. \tag{11}$$