A random walk through number theory

Roland van der Veen

Prime numbers

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2545454542545245245245452478999000896532
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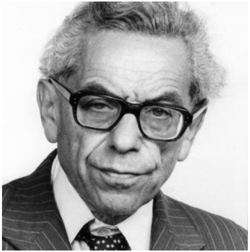
Are prime numbers random?



God may not play dice with the universe,
but something strange is going on with the prime numbers

– Paul Erdös

A probabilistic approach to prime number theory

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4. Random walk implies Riemann hypothesis

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Here k is the number of distinct prime factors of n.

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Conclusion (Prime Number Theorem):

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Moreover, this method allows you to predict the theorems/conjectures of the future!

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Just add squares everywhere in the previous computation.

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Defintion (Riemann zeta function $\zeta(z)$):

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \quad \zeta(2) = \frac{\pi^2}{6}$$

2. Riemann hypothesis

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In view of the formula

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} = \prod_{p \;\; prime} \frac{1}{1 - \frac{1}{p^z}}$$

This statement has a profound impact on the behaviour of the prime numbers.

2. Riemann hypothesis attack plan

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to show that $\frac{1}{\zeta(z)} < \infty$ for $Re(z) > \frac{1}{2}$

3. Möbius function recall

The Möbius function $\mu(n)$ is defined by

$$\mu(n) = \begin{cases} (-1)^k & \text{if } n \text{ is square free} \\ 0 & \text{if } n \text{ is not square free} \end{cases}$$

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For example $\mu(2) = -1$ $\mu(6) = 1$ $\mu(30) = -1$
$\mu(28) = 0$ $\mu(200) = 0$ (and $\mu(1) = 1$ by definition)

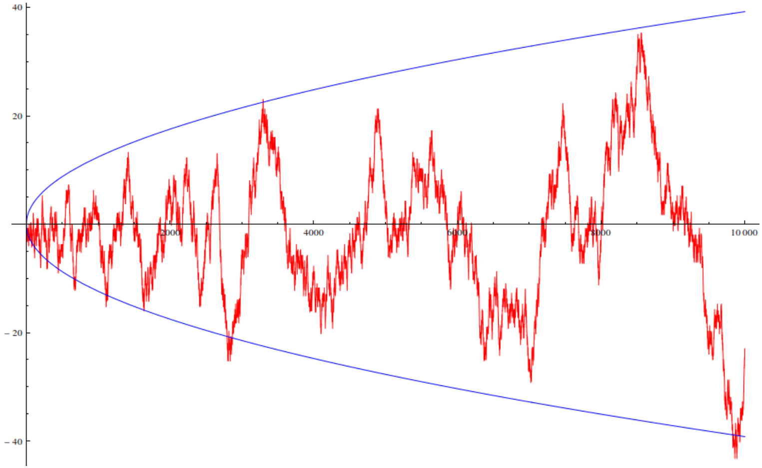
3. The Möbius random walk

Since $\mathbb{P}$(n is square free) $= \frac{6}{\pi^2}$ let's assume $\mu(n)$ behaves randomly as follows:

$$\mu(n) = \begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } \frac{6}{\pi^2} \\ -1 & \text{with probability } p \end{cases}$$

Then the sum $M(x) = \sum_{n=1}^{x} \mu(n)$ is a random walk.

3. Graph of the function $M(x)$



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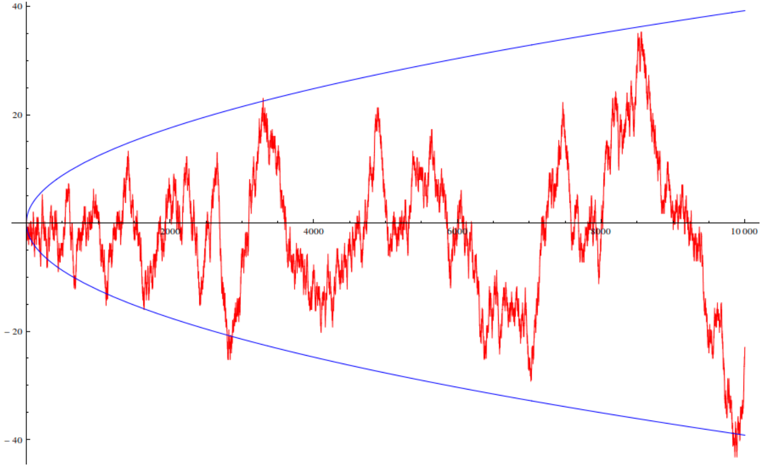
so if the $\mu(n)$ are independent then the Central Limit Theorem says:

$$\sigma M(x) = \sqrt{2px} = \sqrt{x(1 - \frac{6}{\pi^2})}$$

3. Graph of the function $\sqrt{x(1-\frac{6}{\pi^2})}$



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$$\left[\frac{M(x)}{x^z} \right]_1^{\infty} + z \int_1^{\infty} \frac{M(x)}{x^{z+1}} dx \le \int_1^{\infty} \frac{\sqrt{x}}{x^{z+1}} dx < \infty$$

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So the Riemann hypothesis follows from our randomness assumptions.

4. Möbius randomness implies Riemann

Actually the estimate $M(x) \leq \sqrt{x}$ we used is actually bit too crude. It is more appropriate to use the **Law of the Iterated Logarithm** which says that with probability 1 we have

$$M(x) < \sqrt{2\sigma^2 x \log\log(x)}$$

In this way we get a more credible derivation of the Riemann hypothesis (but still no proof of course).

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How to properly apply probability theory to deterministic but apparently random situations?