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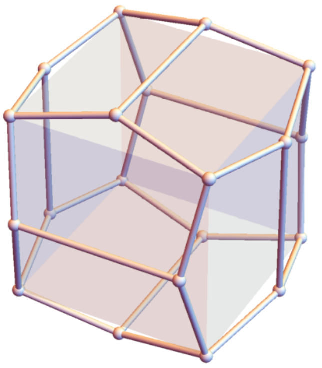
Geometry



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Chapter 0

Introduction

What is geometry? Is it making explicit those parts of our reasoning that deal with how we function in space? How we find our way home and manage to hit things with a stick? Or is it the language of the universe itself? A more mathematical way to approach this question is to focus on the motions that leave all our geometric objects and statements invariant those are the rigid motions or *isometries*[1]. Following Felix Klein we are rather free to decide what kind of transformations on a space make up our isometries. All Klein required was that we have a group G of transformations acting on a set X. Any group will do and we will think of G as the isometries of our space X. Any statement about elements in X that is invariant under the action of G will be considered to be part of the geometry of X.

For example some pairs (X, G) we will meet in this course are:

1. (Set theory) X a set, G all bijections from X to itself.

2. (Topology) X a topological space, $G =$ all homeomorphisms from X to itself.

3. (Affine geometry) X a vector space, $G =$ the group generated by translations and invertible linear maps.

4. (Euclidean geometry) X a vector space with inner product and $G =$ the group generated by translations inner product preserving linear maps.

5. (Projective geometry) X the space of all lines through the origin in a vector space, $G =$ invertible linear maps up to a scalar.

6. (Hyperbolic geometry) X the upper half plane $\{z \in \mathbb{C} | \Im(z) > 0\}$, $G = \{z \mapsto \frac{az+b}{cz+d} | ad - bc = 1\}$.

7. (Spherical geometry) X the unit sphere, G all rotations.

8. (Differential geometry) X a Riemannian chart, G all Riemannian isometries of X to itself.

The list goes on and of course not all of the above terms will make complete sense yet. However the unifying role of the concept of the isometry group G should be apparent. Our focus will not be on topology and set theory but rather on the richer geometries that build on top of those.

Roughly speaking the material in these notes can be divided into a part on algebraic geometry and a part on differential geometry. For both parts but especially for the differential geometry part the reader is supposed to have taken Multivariable analysis with differential forms. Indeed part of the point of this course is to strengthen and illustrate the techniques and ideas introduced in Multivariable analysis.

In the algebraic geometry part of the course (multi)linear algebra will be the workhorse. At first we consider Euclidean geometry, focusing on isometries and simplices (triangles). Next we make an excursion into projective geometry.

In the differential geometry part of the course we do local Riemannian geometry. Concretely this means we have a notion of inner product between vectors at any point but the inner product may vary continuously throughout the space. For simplicity we will mostly focus on the two-dimensional case. This will lead to a discussion of curvature and the famous Gauss-Bonnet theorem. At the end we move back to arbitrary dimensions and discuss how to set up a notion of Laplacian for differential forms. This provides a first excursion into geometric partial differential equations.

[1] Be careful! this word is used differently from what you may be used to in metric spaces!

0.1 Linear algebra review

In this section we briefly recall some abstract notions in linear algebra that will be the foundation of our geometry discussion. For a more detailed account see the excellent book Linear algebra by Klaus Jaënich and the lecture notes on multivariable analysis.

By V we usually mean an n-dimensional vector space. A map $L : V \to W$ is called linear if $\forall v, w \in V,\ \forall a \in \mathbb{R} :\ L(av + w) = aL(v) + L(w)$. The identity $\mathrm{id}_V : V \to V$ defined by $\forall v \in V :\ L(v) = v$ is an example of a linear map and so are shears, rotations and reflections as we will see below. The set of linear maps from V to W is denoted by $\mathrm{Hom}(V, W)$. When $W = \mathbb{R}$ we use the special name dual space $V^* = \mathrm{Hom}(V, \mathbb{R})$. For any $L \in \mathrm{Hom}(V, W)$ we have a pull-back or dual map $L^* \in \mathrm{Hom}(W^*, V^*)$ defined by $L^*f = f \circ L$.

An important class of linear maps are the shears.

Definition 0.1.1. (Shear transformation)
For any $0 \neq \phi \in V^*$ and $0 \neq s$ satisfying $\phi(s) = 0$ define the **shear** *transformation $S_{s,\phi} \in \mathrm{Hom}(V, V)$ by $S_{s,\phi} = \mathrm{id}_V + s\phi$.*

A basis of V is really a linear isomorphism $\mathbf{b} \in \mathrm{Hom}(\mathbb{R}^k, V)$ and that we often use the abbreviation $b_i = b(e_i)$, where $e_i = (0, 0, \ldots 1, \ldots 0)$ denotes the i-th standard basis vector of $\mathbb{R}^n$ with all zeros except a 1 at the i-th place. A basis β of V^* is said to be the dual basis to basis $\mathbf{b}$ of V if $\beta^i(b_j) = \delta^i_j$.

Recall the exterior algebra can be thought of as the algebra of generalized vectors. Instead of just vectors we also allow wedge products of vectors. Roughly speaking we can think of a wedge product $v_1 \wedge v_2$ of vectors v_1, v_2 as representing an parallelpiped spanned by $v_1[0, 1] + v_2[0, 1] = \{x \in V : x = a_1v_1 + a_2v_2,\ a_1, a_2 \in [0, 1]\}$. Algebraically it is convenient to allow arbitrary linear combinations of such wedge products and so we arrive at the definition of exterior algebra:

Definition 0.1.2. (Exterior algebra)
The exterior algebra ΛV is a vector space that contains V as a linear subspace and also contains a nonzero element $\mathbf{1} \notin V$ called the unit. ΛV comes with a map $\wedge : \Lambda V \times \Lambda V \to \Lambda V$ called the wedge product. Using the short hand $\bigwedge_{i=1}^k v_i = v_1 \wedge v_2 \wedge \cdots \wedge v_k$ define $\Lambda^0 V = \mathrm{Span}\{\mathbf{1}\}$, $\Lambda^1 V = V$ and $\Lambda^k V = \mathrm{Span}\{\bigwedge_{i=1}^k v_i | v_i \in V\}$. The wedge product has the following properties:

1. $v \wedge w = -w \wedge v \quad \forall v, w \in V$ (antisymmetry)
2. $(ax_1 + x_2) \wedge y = a(x_1 \wedge y) + (x_2 \wedge y) \quad \forall x_1, x_2, y \in \Lambda V,\ \forall a \in \mathbb{R}$
   $y \wedge (ax_1 + x_2) = a(y \wedge x_1) + (y \wedge x_2)$ (bilinearity)
3. $(x \wedge y) \wedge z = x \wedge (y \wedge z),\ \forall x, y, z \in \Lambda V$ (associativity)
4. $\mathbf{1} \wedge x = x = x \wedge \mathbf{1} \quad \forall x \in \Lambda V$ (unit)
5. *If $v_1, \ldots, v_n \in V$ are independent then* $\bigwedge_{i=1}^n v_i \neq 0$ (basis)
6. *For all $x \in \Lambda V$ there exist unique $x_i \in \Lambda^i V$ such that $x = \sum_{i=0}^n x_i$* (direct sum)

Without proof we mention how one can find a basis for $\Lambda^k V$ using a basis for V.

Lemma 0.1.1. (Basis for $\Lambda^k V$)
For any fixed k a basis $b_1, \ldots b_n$ for V gives a basis of $\Lambda^k V$ consisting of all elements $b_I = \bigwedge_{i \in I} b_i$ where I is an increasing sequence of length k whose members are from $\{1, \ldots, n\}$. The set of basis vectors b_I is ordered lexicographically. Consequently $\dim \Lambda^k V = \binom{n}{k}$.

For example $\Lambda^2\mathbb{R}^4$ has basis $e_{12}, e_{13}, e_{14}, e_{23}, e_{24}, e_{34}$, where $e_{ij} = e_i \wedge e_j$. In the special case $k = n$ we get $\dim \Lambda^n V = 1$ with single basis vector $b_{1\ldots n}$. This is a very important element as it is used to define volume and determinant.

Shears are intimately related to exterior algebra and the following lemma tells us that we should really be considering parallelepipeds up to shears.

Lemma 0.1.2. (Shear equivalence)
First if $v_1, \ldots v_k \in V$ are linearly dependent then $\bigwedge_{i=1}^k v_i = 0$.
Now imagine two sequences of k independent vectors $v_1, \ldots v_k$ and $w_1 \ldots w_k$ in V. The following are equivalent:

1. $\bigwedge_{i=1}^k v_i = \bigwedge_{i=1}^k w_i$
2. $\mathrm{Span}\{v_1, \ldots v_k\} = \mathrm{Span}\{w_1, \ldots w_k\} = U \subset V$ *and the corresponding parallelepipeds $\sum_{i=1}^k [0, 1]v_i$ and $\sum_{i=1}^k [0, 1]w_i$ are related by a finite sequence of shears in U.*

A linear map $L : V \to W$ induces a linear map between wedge powers, simply because it allows us to map parallelepipeds in V to parallelepipeds in W.

Definition 0.1.3. (The induced map $\Lambda^k L$)
Given $L \in \mathrm{Hom}(V, W)$ define the **induced map** $\Lambda^k L \in \mathrm{Hom}(\Lambda^k V, \Lambda^k W)$ *by $(\Lambda^k L) \bigwedge_{i=1}^k v_i = \bigwedge_{i=1}^k L(v_i)$ for all $v_i \in V$ and extending linearly in the sense that $(\Lambda^k L)(ax + y) = a(\Lambda^k L)(x) + (\Lambda^k L)(y)$ for all $x, y \in \Lambda^k V$ and $a \in \mathbb{R}$. For $k = 0$ we set $\Lambda^0 L(\mathbf{1}) = \mathbf{1}$.*

Next we recall a nice definition of the determinant using our formalism.

Definition 0.1.4. (determinant)
The **determinant** $\det L$ *of a linear map $L \in \mathrm{Hom}(V, V)$ is defined by $\Lambda^n L = \det(L)\mathrm{id}_{\Lambda^n V}$. For a sequence of vectors $v_1, \ldots v_n \in \mathbb{R}^n$ we also set $\det(v_1, \ldots, v_n) = \det L$ where $L \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^n)$ is defined as $Le_i = v_i$ for all i.*

Finally we recall how elements of $\Lambda^k(V^*)$ can be evaluated on elements of $\Lambda^k V$ or said differently, how they correspond to dual vectors.

Lemma 0.1.3. (Dual exterior equals exterior dual)
The linear map $\mathcal{I} : \Lambda^k(V^*) \to (\Lambda^k V)^*$ defined by

$$\big(\mathcal{I}(\bigwedge_{j=1}^k \phi_j)\big)(\bigwedge_{i=1}^k v_i) = \det(\phi(v))$$

and extended linearly is an isomorphism. Here $(\phi(v))$ is the $k \times k$ matrix whose columns are $\phi^1(v), \ldots \phi^k(v)$.

Combining the induced map with the dual of $L : V \to W$ we get the pull-back $L^* : \Lambda^k(W^*) \to \Lambda^k(V^*)$ defined as $L^*\omega = \Lambda^k L^*$.

Chapter 1

Euclidean geometry

In this chapter we will use linear algebra to discuss Euclidean geometry. For dealing with angles and distances we make use of an inner product so first we recall some basic notions about inner product spaces. On the other hand if all we want to do is discuss intersections of lines and planes then no inner product is needed and for such investigations affine geometry provides an appropriate language. After giving some basic definitions about affine geometry we turn to Euclidean isometries and show how they can be understood using reflections. Next is a small taste of plane geometry and finally we briefly explore simplices and simplicial complexes in Euclidean space, culminating in the Euler characteristic.

1.1 Affine geometry

Linear algebra provides a very convenient language for discussing the intersection of lines and planes. However the choice of origin is not natural from a geometric point of view. Geometrically there is no reason why the x-axis in the plane $\mathbb{R}^2$ is more special than any other line parallel to it. This brings us to the notion of affine subspace, where we allow ourselves to translate the origin to a different location. Affine geometry is the geometry of vector space where we ignore the origin or said differently allow arbitrary translations.

Definition 1.1.1. (Affine subspace)
Suppose $U \subset V$ is a linear subspace and $w \in V$ fixed. The subset $U + w = \{u + w | u \in U\} \subset V$ is called an **affine subspace** *of V. If $\dim(U) = 1$ we say that $U + w$ is a line. The dimension of $U + w$ is the dimension of U and the codimension of $U + w$ is $\dim(V) - \dim(U)$. A subspace of codimension 1 is called a hyperplane.*

Just like with linear independence there is a notion of affine independence.

Definition 1.1.2. (Affine independent)
A subset $S \subset V$ is said to be **affine independent** *if for all $s \in S$ the set $\{t - s | t \neq s\}$ is linearly independent.*

Coming back to our example of the x-axis X in $V = \mathbb{R}^2$, we have $X = \mathrm{Span}(e_1)$, a linear subspace. An example of an affine subspace of $\mathbb{R}^2$ would be $P = X + e_2$, a line parallel to the x-axis. Incidentally the lines in $\mathbb{R}^2$ are also hyperplanes, in $\mathbb{R}^3$ the hyperplanes are just planes.

For any two distinct points $v \neq w \in V$ there exists a unique line passing through both v, w. This line can be presented as $L = \mathrm{Span}(v - w) + w$. Suppose we had another line $M = U + w$ containing v, w then $L \subset M$. Since U has dimension 1 we must have $U = \mathrm{Span}(v - w)$ and so $L = M$. The line through distinct points A, B is often denoted AB.

Affine subspaces are the images of linear subspaces under translations and for the record we make explicit what we mean by a translation:

Definition 1.1.3. (Translation map)
A **translation** *by a vector w is a map $T_w : V \to V$ sending $T_w(v) = v + w$.*

In our planar example we have $P = T_{e_2}(X)$. The geometry that appears when we augment linear algebra by translations is called *affine geometry*. The reader should be careful that translations are never[1] linear maps because the origin is moved: $T_w(0) = w$. More generally the maps that preserve the structure of affine geometry are called affine maps.

Definition 1.1.4. (Affine map)
For vector spaces V, W we say $f : V \to W$ is an affine map if it is the composition of (finitely many) linear maps and translations. The affine group $GA(V)$ consists of all invertible affine maps from V to V.

[1] unless $w = 0$ and they are the identity of course

An example of an affine map is a dilation $D_{c,\lambda} : V \to V$ with center $c \in V$ and factor $\lambda \in \mathbb{R}$ defined as $D_{c,\lambda}(v) = \lambda(v - c) + c$. When $\lambda \neq 0$ the dilation $D_{c,\lambda}$ is an example of an element of the affine group $GA(V)$. Translations are also in $GA(V)$.

Affine maps may not be linear but they behave like linear maps when we apply them to a difference of two vectors. In a difference between two vectors the choice of origin disappears or said differently translations have no effect so what is left are linear maps. We will sometimes use the notation $\overrightarrow{ab} = b - a$ for $a, b \in V$. Then we claim (exercise!) that for any affine map $F : V \to V$ there exists a linear map $L \in \mathrm{Hom}(V, V)$ such that $F(\overrightarrow{ab}) = L(\overrightarrow{ab})$ for all $a, b \in V$.

We cannot speak about distances in affine geometry. However the ratio between two line segments does make sense (exercise!). It also is meaningful to talk about parallel lines or affine subspaces.

Definition 1.1.5. (Parallelism)
Two affine subspaces $A, B \subset V$ are parallel if $A = T_v(B)$ for some $v \in V$.

Parallel lines can sometimes be inconvenient special situations. In the next chapter on projective geometry we will simplify the situation by adding points at infinity to make sure all lines intersect in a point and the notion of parallelism disappears.

Perhaps surprisingly ratios between lengths of segments make sense in affine geometry. More precisely, if $a, b, c, d \in V$ satisfy $a \neq b$ and $c \neq d$ and the lines ab and cd parallel, then we denote by $\lambda = \frac{\overrightarrow{ab}}{\overrightarrow{cd}}$ the unique real number such that $a - b = \lambda(c - d)$. The number λ is invariant under affine transformations in the sense that

$$\frac{\overrightarrow{F(a)F(b)}}{\overrightarrow{F(c)F(d)}} = \frac{\overrightarrow{ab}}{\overrightarrow{cd}} \tag{1.1}$$

As a sample theorem in plane affine geometry we consider Ceva's theorem.

Theorem 1.1.1. (Ceva)
Assume $\dim V = 2$ and suppose $A_0, A_1, A_2 \in V$ are three points not on a line. If X is a point not on any line A_iA_j where all three lines A_0X, A_1X, A_2X meet then

$$\frac{\overrightarrow{A_1B_0}}{\overrightarrow{B_0A_2}} \cdot \frac{\overrightarrow{A_2B_1}}{\overrightarrow{B_1A_0}} \cdot \frac{\overrightarrow{A_0B_2}}{\overrightarrow{B_2A_1}} = 1$$

where $B_i = A_iX \cap A_jA_k$ and $\{i, j, k\} = \{0, 1, 2\}$.

We will prove it using the wedge product. The wedge product enters the game through the following lemma:

Lemma 1.1.1. (Wedge product ratios)
If $a, b, c \in V$ are distinct points on an affine line not through the origin, then

$$\frac{\overrightarrow{ab}}{\overrightarrow{bc}} = \frac{\overrightarrow{a \wedge b}}{\overrightarrow{b \wedge c}}$$

Proof. The later ratio makes sense because b and c must be linearly independent (Exercise!). Since a is distinct from b, c we must have non-zero numbers $\lambda, \mu \in \mathbb{R}$ such that $a = \lambda b + \mu c$. It follows that $a \wedge b = -\mu b \wedge c$ so $\frac{\overrightarrow{a \wedge b}}{\overrightarrow{b \wedge c}} = -\mu$. On the other hand if we set $\sigma = \frac{\overrightarrow{ab}}{\overrightarrow{bc}}$ then $\sigma(c - b) = b - a = b - \lambda b - \mu c$. Taking the coefficient of c proves $\sigma = -\mu$. ☐

The proof of Ceva's theorem now goes as follows. By applying a translation if necessary we may assume that $X = 0$ and so there are non-zero λ_i such that $B_i = \lambda_i A_i$. Also we know that A_1, A_2 are linearly independent so that $A_0 = \alpha_1 A_1 + \alpha_2 A_2$ for some non-zero α_1, α_2. Lemma 1.1.1 tells us that if i, j, k are distinct we have

$$\frac{\overrightarrow{A_iB_j}}{\overrightarrow{B_jA_k}} = \frac{A_i \wedge B_j}{B_j \wedge A_k} = \frac{A_i \wedge A_j}{A_j \wedge A_k}$$

When $(i, j, k) = (1, 0, 2)$ this ratio is $\frac{\alpha_2}{\alpha_1}$ since $A_1 \wedge A_0 = A_1 \wedge (\alpha_1 A_1 + \alpha_2 A_2) = \alpha_2 A_1 \wedge A_2$ and $A_0 \wedge A_2 = (\alpha_1 A_1 + \alpha_2 A_2) \wedge A_1 = \alpha_1 A_1 \wedge A_2$. For the same reason when $(i, j, k) = (2, 1, 0)$ the ratio is $-\alpha_2^{-1}$ and when $(i, j, k) = (0, 2, 1)$ the ratio is $-\alpha_1$ Taking the product of the ratios finishes the proof.

Exercises

1. Prove that two (affine) lines in V can intersect in $0, 1$ or infinitely many points.
2. By a plane we mean a two-dimensional affine subspace of V. Is it true that there is a unique plane passing through any three distinct points $A, B, C \in V$? If yes give a proof, if no show a counter example and formulate an additional (linear algebraic) condition on A, B, C under which the statement is true.
3. In this exercise we prove a version of Pappus' theorem called parallel Pappus. We say three points are colinear if they are contained in the same line. We work in two dimensions so $n = \dim(V) = 2$. Consider two pairs of colinear triples A, B, C and A', B', C' of distinct points in V. If the lines AB' and $A'B$ are parallel and the lines AC' and $A'C$ are parallel, then so are the lines BC' and $B'C$.
   (a) Draw a picture of the six points and the relevant lines.
   (b) Explain why the statement of the parallel Pappus theorem is invariant under translations and more generally affine transformations.
   (c) Prove the theorem under the assumption that B and B' are on the same line.
   (d) Now assume that A is the origin and B, B' form a basis of V. What are the coordinates of A' and C, C' with respect to this basis?
   (e) Prove the parallel Pappus theorem in general.
4. Show that any linear subspace $U \subset V$ of dimension 2 corresponds to a unique line through the origin in $\Lambda^2(V)$.
5. Consider two affine subspaces $U \subset V$ and $J \subset W$ and an affine map $F : V \to W$. Prove that the image $F(U)$ and the inverse image $F^{-1}(J)$ are both affine subpaces.
6. Under what conditions is the composition of the dilations $D_{c,\lambda}$ and $D_{c',\lambda'}$ again a dilation? Can the composition also be a translation?
7. Show that points $A, B, C \in V$ are colinear (belong to the same affine line) if and only if $A \wedge B + B \wedge C + C \wedge A = 0$.
8. Lemma 1.1.1 is not clearly a statement in affine geometry in the sense that it mentions the origin. Can you reformulate the lemma to make it a truely affine geometry statement? Hint: Replace the origin by an arbitrary point not on the line.
9. For any $S \subset V$ we denote by $\langle S \rangle$ the intersection of all affine subspaces containing S. Show that $\langle S \rangle$ is an affine subspace. Prove that a set S is affine independent if and only if for all $s \in S$ we have $\dim\langle S - s \rangle < \dim\langle S \rangle$.

1.2 Vector spaces with an inner product

In order to discuss distances and angles in Euclidean geometry we need to add more structure to our vector space. A convenient way to set up this structure is by introducing an inner product (dot product) in V.

Definition 1.2.1. (inner product)
By an inner product on V we mean a bilinear map $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ satisfying

1. $\forall v, w \in V : \ \langle v, w \rangle = \langle w, v \rangle$ *(symmetry)*
2. $\forall v \in V \ (\forall w \in V : \langle v, w \rangle = 0) \Rightarrow v = 0$ *(non-degeneracy)*
3. $\forall v \neq 0 : \ \langle v, v \rangle > 0$ *(positivity)*

We choose not to just work with $\mathbb{R}^n$ with the standard inner product $\langle \sum_{i=1}^n v^i e_i, \sum_{j=1}^n w^j e_j \rangle = \sum_{i=1}^n v^i w^i$. The reason for our more abstract approach is twofold. First working with matrices and a fixed basis is more cumbersome. Second, in the chapter on Riemannian geometry we will allow the inner product to vary as we move through our curved space.

Recall the norm corresponding to the inner product is $|v| = \sqrt{\langle v, v \rangle}$ where we take the non-negative square root.

The notion of perpendicular is very important, recall $v \perp w$ if $\langle v, w \rangle = 0$. For any subset $S \subset V$ we define $S^\perp = \{v \in V : \forall s \in S, \ \langle s, v \rangle = 0\}$. When S is a linear subspace of V then $V = S \oplus S^\perp$. Recall that $A = B \oplus C$ means every element a of vector space A can uniquely be written as $a = b + c$ for some b in subspace $B \subset A$ and c in subspace $C \subset A$.

An orthonormal basis $\mathbf{b}$ of V means that $\langle b_i, b_j\rangle = \delta_{ij}$ for all $i, j = 1, \dots n$. One of the useful things about an orthonormal basis is that for any $v \in V$ we can write $v = \sum_{i=1}^{n}\langle v, b_i\rangle b_i$. In other words, the coefficient of v with respect to b_i is just the their dot product.

The Gram-Schmidt algorithm allows us to transform any basis into a related orthonormal basis. To state it concisely we use the notation $N(v) = \frac{v}{|v|}$ for any $v \neq 0$. Suppose we have any basis $\mathbf{b}$ then define the new basis $\mathbf{c}$ inductively by setting

$$c_k = N(b_k - \sum_{i=1}^{k-1}\langle b_k, c_i\rangle c_i)$$

When $k = 1$ the sum is taken to be 0 by convention. It should be clear that the c_i are linear combinations of the b_i and that they have norm 1. The reader should check that $\langle c_i, c_j\rangle = \delta_{ij}$.

The inner product sets up a linear isomorphism $\Xi : V \to V^*$ defined by $\Xi(v)(w) = \langle v, w\rangle$.

The sphere in V with radius r and center p is called $S_r(p) = \{v \in V : |v - p| = r\}$.

Recall that a matrix M is positive definite if $v^T M v > 0$ whenever $v \neq 0$.

Lemma 1.2.1. *If $\mathbf{b}$ is a basis of V then the matrix M defined by $M_{ij} = \langle b_i, b_j\rangle$ is positive definite and $\langle v, w\rangle = \mathbf{b}^{-1}(v)^T M\mathbf{b}^{-1}(w)$. Conversely, given any positive definite matrix M the previous formula defines an inner product.*

Proof. The equation for $\langle v, w\rangle$ follows from expressing v, w in the basis $\mathbf{b}$ and using bilinearity of the inner product. The positive definiteness of the matrix is then equivalent to the positive definiteness of the inner product by definition. □

Exercises

1. Prove the Cauchy-Schwarz inequality $|\langle v, w\rangle|^2 \leq \langle v, v\rangle\langle w, w\rangle$.
2. Run the Gram-Schmidt algorithm to construct an orthonormal basis of the subspace of $V = \mathbb{R}^4$ generated by e_1 and $e_1 + e_4$. Here we take $\mathbb{R}^4$ with the usual dot product.
3. Does the formula $\langle(a, b), (c, d)\rangle = ac + 2ad + 3bc + 4bd$ determine an inner product on $\mathbb{R}^2$? If so find an orthonormal basis.
4. Show that for any unit vector u the vector $v - u\langle v, u\rangle$ is in $\{v\}^\perp$.
5. Is it always true that $v + w \perp v - w$ or do you need additional assumptions on v, w?

1.3 Euclidean space and its isometries

Unlike Euclid's axiomatic approach we prefer to define Euclidean space in vector space language. The theory is equivalent but much more convenient to calculate and prove things. Everything we said about affine geometry still applies but of course Euclidean space allows for more refined constructions using distances.

Definition 1.3.1. (Euclidean space)
By a Euclidean space we mean an n-dimensional vector space V together with an inner product $\langle ., .\rangle$. The distance between two points A, B is the norm $|A - B|$.

One way to study geometry is by studying the motions that preserve all geometric objects and properties. These are called the isometries. In our Euclidean vector space the isometries that fix the origin are the linear maps that preserve the inner product. Since they must preserve orthogonality they are known as the orthogonal transformations. A better name however would be orthonormal transformations since they also preserve the norm of any vector.

Definition 1.3.2. (Orthogonal transformations)
An element $M \in \mathrm{Hom}(V, V)$ is said to be orthogonal if it preserves the inner product: $\langle Mv, Mw\rangle = \langle v, w\rangle$. The set of all orthogonal transformations is denoted $O(V)$.

Simple examples of orthogonal transformations are id_V and $-\mathrm{id}_V$. The latter is an orthogonal map because $\langle -v, -w\rangle = \langle v, w\rangle$. More interesting examples of orthogonal transformations are reflections and rotations. We start with reflections since they are easier to understand.

Definition 1.3.3. (Reflection and rotation)
Given a unit vector $m \in V$ we define the **reflection** *$R_m \in O(V)$ by $R_m m = -m$ and $R_m v = v$ for all $v \perp m$. The composition of two reflections is called a* **rotation**.

The vector m is the normal vector to the hyperplane $m^\perp$ that acts as the mirror. An explicit formula for the reflection is $R_m v = v - 2m\langle m, v\rangle$. The reader should check that any reflection is its own inverse and has determinant -1. Perhaps less expected is that reflections generate all orthogonal transformations and thus relate all orthonormal bases.

Theorem 1.3.1. (Reflections generate $O(V)$)
Any two orthonormal bases of V are related by an element of $O(V)$. Furthermore, if for $L \in O(V)$ there exists a linear subspace $U \subset V$ of codimension c such that $\forall u \in U : L(u) = u$, then L is the composition of at most c reflections.

Proof. Given orthonormal bases $\mathbf{b}, \mathbf{c} : \mathbb{R}^n \to V$ consider $L = \mathbf{c} \circ \mathbf{b}^{-1} \in \mathrm{Hom}(V, V)$. The transformation L relates the bases in that $Lb_i = c_i$ for all i. To check that $L \in O(V)$ we only need to check on the basis elements b_i and here it follows from the orthogonality of $\mathbf{c}$ because $\langle Lb_i Lb_j\rangle = \langle c_i, c_j\rangle = \delta_{ij}$ as required.

For the last statement we argue by induction on the codimension $c = n - k$, where $n = \dim V$. When $n - k = 0$ we must have $L = id_V$. For the induction step, suppose L fixes a k-dimensional subspace U and $Lv = w \neq v$. Then $R_{\frac{v-w}{|v-w|}} \circ L$ fixes a $k+1$-dimensional subspace spanned by v and U. This is because $v - w$ is orthogonal to U (why?) so by the induction hypothesis the proof is complete. □

Conjugating by a reflection has the same effect as reflecting the unit vector defining the mirror itself. More precisely: $R_w R_m R_w = R_{R_w(m)}$. To check this is true it suffices to say that both sides send $R_w(m)$ to its negative and both sides fix its orthogonal complement.

Now that we understand orthogonal maps in terms of reflections we can apply this to study the *volume element* $b_{1\ldots n} = b_1 \wedge b_2 \wedge \cdots \wedge b_n \in \Lambda^n V$. We think of that element as the standard unit box in the space V. If $\mathbf{b}$ is a basis then $b_{1\ldots n}$ actually generates $\Lambda^n V$ and the determinant of a linear map $L : V \to V$ can be understood as the scale factor of the induced map $\Lambda^n L$. The absolute value of the determinant tells us how much our box $b_{1\ldots n}$ gets enlarged when we apply L to all its sides. The sign of the determinant is more subtle and has to do with the concept of orientation. As we will see now, a reflection will reverse the orientation of the box.

Lemma 1.3.1. (Volume and orientation)
For any two orthonormal bases $\mathbf{b}, \mathbf{c}$ we have $c_{1\ldots n} = (-1)^r b_{1\ldots n} \in \Lambda^n V$, where r is the number of reflections necessary to pass from $\mathbf{b}$ to $\mathbf{c}$. It follows that the determinant of any reflection is -1.

Proof. If $\mathbf{c} = R_m \circ \mathbf{b}$ for some reflection R_m then write $m = \sum_i m^i b_i$, where $|m|^2 = 1$.

$$c_{1\ldots n} = \bigwedge_{i=1}^{n} R_m b_i = \bigwedge_{i=1}^{n} \left(b_i - 2mm^i\right) = b_{1\ldots n} + \sum_{j=1}^{n} \bigwedge_{i=1}^{n} z_{j,i}$$

where[2] we expanded using $m \wedge m = 0$ and set $z_{j,i} = b_i$ if $i \neq j$ and $z_{i,i} = -2mm^i$. Expanding m in the $\mathbf{b}$ basis and using $b_i \wedge b_i = 0$ we conclude that

$$c_{1\ldots n} = b_{1\ldots n} - 2\sum_{i=1}^{n} (m^i)^2 b_{1\ldots n} = (1 - 2|m|^2) b_{1\ldots n} = -b_{1\ldots n}$$

The general case now follows from Theorem 1.3.1. □

In general orientation is defined on V as follows.

Definition 1.3.4. (Orientation)
Define an equivalence relation on the set of all bases of V by requiring that two bases $\mathbf{b}, \mathbf{c}$ of V are equivalent if $\det \mathbf{c}^{-1} \circ \mathbf{b} > 0$. *An* **orientation** *of V is a choice of an equivalence class of bases. $L \in \mathrm{Hom}(V, V)$ is said to be orientation preserving/reversing if* $\det L$ *is positive/negative.*

The above lemma shows that reflections are orientation reversing maps. We sometimes write $O^+(V)$ for the orientation preserving elements of $O(V)$ and $O^-(V)$ for the orientation reversing ones. Note that $O^+(V)$ must be a subgroup of $O(V)$ and that it is the subgroup generated by all rotations in $O(V)$. Not every element of $O^+(V)$ is a rotation however, unless $\dim V \leq 3$. The special case where $\dim(V) = 2$ is especially important and helps us to make sense of rotations

Lemma 1.3.2. *Suppose* $\dim V = 2$. *Taking the matrix with respect to an orthornomal basis* $\mathbf{b}$ *gives an isomorphism of groups*

$$\Phi_{\mathbf{b}} : O^+(V) \to U = \{\begin{pmatrix} a & -b \\ b & a \end{pmatrix} : a, b \in \mathbb{R},\ a^2 + b^2 = 1\}$$

If $\mathbf{c}$ is another orthonormal basis with the same orientation then $\Phi_{\mathbf{c}} = \Phi_{\mathbf{b}}$. Otherwise $\Phi_{\mathbf{c}} = \left(\Phi_{\mathbf{b}}\right)^{-1}$

[2] you may want to first do this calculation in the case $n = 2$ to see what's going on!

Proof. With respect to an orthonormal basis $\mathbf{b}:\mathbb{R}^2\to V$ we can describe $L\in O^+(V)$ as $\mathbf{b}^{-1}\circ L\circ\mathbf{b}$, whose matrix with respect to the standard basis must have orthonormal columns and determinant 1. If we denote such a matrix by $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ then we must have $a^2+b^2=1=c^2+d^2=ad-bc$ and $ac+bd=0$. It follows that $c=acd-bc^2=-bd^2-bc^2=-b(c^2+d^2)=-b$ and so $1=ad+b^2=ad+1-a^2$ means $a(d-a)=0$ so $a=d$.

Then $\mathbf{b}$ gives us a map $\Phi_\mathbf{b}:O^+(V)\to U$. The map $\Phi_\mathbf{b}$ is a group homomorphism because composition of linear maps corresponds to multiplication of the corresponding matrices. This shows that $O^+(V)$ is commutative because U is. $\Phi_\mathbf{b}$ is injective because the matrix determines the linear map. Finally it is surjective because for any two unit vectors $u,v\in V$ we can find an element of $O^+(V)$ that maps u onto v (Exercise!). So if we want to find $L\in O^+(V)$ such that $\Phi_\mathbf{b}(L)=\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ then we can take $u=\mathbf{b}(e_1)$ and $(a,b)=\mathbf{b}^{-1}(v)$.

If $\mathbf{c}$ is another orthonormal basis of the same orientation then $M=\mathbf{b}^{-1}\circ\mathbf{c}\in O^+(\mathbb{R}^2)$. Then $\Phi_\mathbf{c}(L)$ is the matrix of $\mathbf{c}^{-1}\circ L\circ\mathbf{c}=M^{-1}\circ\mathbf{b}^{-1}\circ L\circ\mathbf{b}\circ M=\mathbf{b}^{-1}\circ L\circ\mathbf{b}$ by commutativity of $O^+(\mathbb{R}^2)$, so $\Phi_\mathbf{c}=\Phi_\mathbf{b}$. Finally we can find a basis with opposite orientation by swapping the two basis vectors of $\mathbf{b}$. This permutes both columns and rows of the corresponding matrix and turns it into its inverse. □

The finite subgroups of $O(V)$ are called finite reflection groups. When $n=3$ these describe the symmetries of the Platonic solids, tetrahedron, cube, octahedron, dodecahedron and icosahedron. One may also wonder what happens beyond dimension three. In any dimension generalizations of tetrahedron, cube and octahedron exist and perhaps surprisingly those are all the Platonic solids possible in dimensions five and up. Dodecahedron and icosahedron do not have a straight-forward higher dimensional equivalent. In dimension four the situation is a more interesting in that six types of Platonic solids exist. A counter-part to each of the five in three-dimensions plus another one called the 24-cell. This acciese figures can be understood in terms of the symmetries of the three-dimensional Platonic solids.

Finally we consider more general Euclidean isometries that do not necessarily fix the origin. Recall from group theory conjugation of F by G means GFG^{-1}. Intuitively conjugation means we are still applying F but we changed our viewpoint by G. More concretely, by conjugating by a translation we can move the center of any orthogonal transformation to an arbitrary point. This brings us to our definition of the general Euclidean isometries called the Euclidean group. Notice the similarity to the affine transformations. The only difference is that we only allow orthogonal linear maps.

Definition 1.3.5. (Euclidean group)
The **Euclidean group** *$E(V)$ is the set of all finite compositions of translations and $O(V)$. For any $c\in V$ we define the subgroup $O(V)_c\subset E(V)$ called the orthogonal transformations with center c by $O(V)_c=\{T_c\circ L\circ T_{-c}|L\in O(V)\}$. In particular $T_c\circ L\circ T_{-c}\in O(V)_c$ are called* **affine reflections** *and* **affine rotations** *if L is a reflection/rotation.*

The fundamental relation in the Euclidean group is that if $L\in O(V)$ and T_w a translation, then $L\circ T_w\circ L^{-1}=T_{L(w)}$. To see why this holds, just apply both sides to a vector $v\in V$. This means that we can write any element of $E(V)$ as T_wL for some $L\in O(V)$ and $w\in V$. Indeed for example $L\circ T_w=L\circ T_w\circ L^{-1}\circ L=T_{L(w)}\circ L$. More algebraically we could say that $E(V)$ is the semi-direct product of $O(V)$ by the additive group V, $E(V)\cong V\rtimes O(V)$.

Our perspective on Euclidean geometry is that it consists of all statements about objects in V that are invariant under the action of $E(V)$ on V. This includes the distance between two points, the existence of the regular dodecahedron and so on. In the next section we will consider the case $\dim V=2$ and study angles and triangles.

Exercises

1. Prove that for any two distinct points $A,B\in V$ there is a unique hyperplane perpendicular to the line through A,B and containing the midpoint $(A+B)/2$. Show that it contains all points whose distance to A is equal to the distance to B.

2. For any linear subspace $F\subset V$ define $V\xrightarrow{s_F}V$ by $s_F(x+y)=x-y$ where $x\in F$ and $y\in F^\perp$. Prove that s_F is in $O(V)$. How many reflections do you need to construct s_F? What happens when F is a hyperplane?

3. Prove that any translation is the composition of two affine reflections.

4. Prove that any affine rotation is the composition of two affine reflections.

5. Prove that any element of $E(V)$ is the composition of finitely many affine reflections. How many do you need at most when $\dim(V)=n$?

6. Take $\dim(V) = 2$ and prove that the composition of two affine rotations is again an affine rotation or a translation. Where is the center of the rotation if it exists?

7. For which $\phi \in V^*$ and $s \in \ker \phi$ is the shear $S_{s,\phi}$ a Euclidean isometry? In such cases can you write it as a composition of reflections?

8. Come up with an element $F \in E(\mathbb{R}^2)$ that has no fixed points. A fixed point is a $v \in V$ such that $F(v) = v$. Can you arrange it such that F is NOT a rotation or a translation or a reflection?

9. Consider two unit vectors $u, v \in V$. Prove that $R_{\frac{u-v}{|u-v|}}$ sends u to v and v to u. Conclude that if w is a unit vector perpendicular to u then $R_{\frac{u-v}{|u-v|}} \circ R_w$ is an element of $O^+(V)$ that sends u to v as required in the proof of Lemma 1.3.2.

10. More on the plane orthogonal group. Take $\dim V = 2$
    (a) Explain why $O^+(V)$ is a commutative group.
    (b) For any unit vector $u \in V$ prove that $R_u \in O^-(V)$.
    (c) Verify $O^-(V) \cap O^+(V) = \emptyset$ and $O(V) = O^+(V) \cup O^-(V)$.
    (d) Show why for any element $L \in O(V)^-$ there is a unique $M \in O^+(V)$ such that $R_u \circ M$.
    (e) Choose an orthonormal basis $\mathbf{b}$ for V and write down the matrix for R_{b_1} with respect to this basis.
    (f) Use $\Phi_{\mathbf{b}}$ and extend it to $O(V)$.
    (g) Is $O(V)$ commutative?
    (h) Can you find a copy of the dihedral group D_n inside $O(V)$?

11. In this exercise we show how to determine the result of composing two affine rotations in the plane. Take $\dim(V) = 2$. For any affine line $\ell \subset V$ we denote by ρ_ℓ the affine reflection in line ℓ. So $\rho_\ell = T_c o R_m o T_{-c}$ for any $c \in \ell$ and any unit vector m perpendicular to ℓ.
    (a) Suppose $r \in O(V)$ is a rotation. Prove that $r \circ R_a \circ r^{-1} = R_{r(a)}$.
    (b) Generalize this formula to $r \circ \rho_\ell \circ r^{-1} = \rho_{r(\ell)}$ for any rotation $r \in O_c(V)$ with $c \in \ell$.
    (c) Explain why any affine rotation with center c can be written as $\rho_\ell \circ \rho_\mu$ with $c = \mu \cap \ell$ and μ, ℓ affine lines. Moreover, one of μ, ℓ can be chosen freely among all lines passing through c.
    (d) Consider two affine rotations A and B with centers $a \neq b$. Prove that we may write them as $A = \rho_\gamma \circ \rho_\beta$ and $B = \rho_\alpha \circ \rho_\gamma$ for affine lines α, β, γ such that $\alpha \cap \gamma = b$ and $\beta \cap \gamma = a$.
    (e) Prove that $B \circ A = \rho_\alpha \circ \rho_\beta$.
    (f) Conclude from your formula that $B \circ A$ is a rotation with center $\alpha \cap \beta$ provided that intersection is not empty.
    (g) What happens when the lines α, β do not intersect, i.e are parallel?

12. Spatial rotations. Let us take $\dim(V) = 3$ throughout this exercise.
    (a) Is every element of $O^+(V)$ a rotation?
    (b) Prove that for any non-trivial rotation $\rho \in O(V)$ there exists a unique line A (the axis) such that $\rho(a) = a$ for all $a \in A$.
    (c) Show that the restriction of ρ to $A^\perp$ is a rotation in $O^+(A^\perp)$.
    (d) Pick an orthonormal basis $\mathbf{b}$ of V and consider $\Delta = \{b_1 + b_2 + b_3, b_1 - b_2 - b_3, b_2 - b_3 - b_1, b_3 - b_1 - b_2\}$. Try to find as many elements $f \in O^+(V)$ such that $f(\Delta) = \Delta$ and give for each of them a vector that spans the axis of the rotation.

13. Can you write the map $a : V \to V$ defined by $a(v) = -v$ as a composition of reflections in the n-dimensional inner product space V? If so how many? Is a orientation preserving? Is it in $O(V)$?

14. **Quaternions and three-dimensional rotations**.
    In this exercise we recall the quaternions and make some statements about their relation to Euclidean isometries that the reader is invited to check. Recall the quaternions H is a 4-dimensional vector space with basis $1, i, j, k$ and bilinear, associative product $H \times H \to H$ determined by Hamilton's relations

$$\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = -1 \quad \mathrm{ij} = \mathrm{k} \quad \mathrm{jk} = \mathrm{i} \quad \mathrm{ki} = \mathrm{j}$$

If $q = w + x\mathrm{i} + y\mathrm{j} + z\mathrm{k} \in H$ the conjugate is $\bar{q} = w - x\mathrm{i} - y\mathrm{j} - z\mathrm{k}$. The norm is defined as $|q| = q\bar{q} = w^2 + x^2 + y^2 + z^2 \geq 0$.

Calculus books still use $\mathrm{i}, \mathrm{j}, \mathrm{k}$ to denote the unit vectors in $\mathbb{R}^3$ and this originated from Hamilton's work. If we identify vector $v = (v_1, v_2, v_3) \in \mathbb{R}^3$ with the imaginary quaternion $q_v = v_1\mathrm{i} + v_2\mathrm{j} + v_3\mathrm{k} \in H$ then we have

$$q_v q_u = -v \cdot u + q_{v\times u}$$

This means that vectors $v \perp u$ if and only if $q_v q_u + q_u q_v = 0$ because $\bar{q}_v = -q_v$.

Now assume u is a unit vector so that $q_u \bar{q}_u = 1$. Then $q_u^2 = -1$ and $v \perp u$ if and only if $q_v = q_u q_v q_u$. Notice that the left hand side is again a pure quaternion. This means there is a map $s : \mathbb{R}^3 \to \mathbb{R}^3$ given by $v \mapsto q_u q_v q_u = q_{s(v)}$. We claim $s = R_u$ the reflection in the plane orthogonal to unit vector u. Indeed s fixes vectors in the plane $u^\perp$ by the above discussion and $q_{s(u)} = q_u^3 = -q_u = q_{-u}$ since $q_u^2 = -1$.

For two unit vectors u, t composing the reflections $s_{u^\perp}$ and $s_{t^\perp}$ we get a map sending q_v to $q_t q_u q_v q_u q_t$. Now $q_t q_u = (q_u q_t)^{-1}$ since both u, t are unit vectors. The product $q_t q_u$ is also a unit quaternion so all rotations are written in quaterions as conjugation by a unit quaternion.

As an application of all this, try to realize the symmetries of the 3d Platonic solids as subsets of the unit sphere in H. Argue that these subsets must be 4d Platonic solids in their own right.

1.4 Angles and triangles

In this section $\mathbb{E}$ is a two-dimensional vector space with inner product with a fixed orientation. We say a basis is positive if it is in the chosen orientation. In this case the group of rotations $O^+(\mathbb{E})$ is generated by pairs of reflections in lines through the origin.

We also know that there is an isomorphism using any positive orthonormal basis $\mathbf{b}$.

$$\Phi_{\mathbf{b}} : O^+(V) \to U = \{\begin{pmatrix} a & -b \\ b & a \end{pmatrix} : a, b \in \mathbb{R},\ a^2 + b^2 = 1\}$$

To turn our angles into numbers we use the group isomorphism $\mathcal{A} : U \to \mathbb{R}/2\pi\mathbb{Z}$ whose inverse is $\mathcal{A}^{-1}\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

For any pair of unit vectors u, v there is a unique rotation $L \in O^+(\mathbb{E})$ sending u to v. To see this we extend u to the unique positive orthonormal basis u, w of $\mathbb{E}$. Then v can be expressed as $v = \langle v, u\rangle u + \langle v, w\rangle w$ and then $\Phi_{u,w}^{-1}\begin{pmatrix} \langle v, u\rangle & -\langle v, w\rangle \\ \langle v, w\rangle & \langle v, u\rangle \end{pmatrix}$ sends u to v. No choices were made so this gives a unique element of $O^+(\mathbb{E})$.

In conclusion our concept of angle comes down to the following.

Definition 1.4.1. (Angles)
The angle $\angle(u, v)$ between two non-zero vectors in $u, v \in \mathbb{E}$ is defined in terms of the unique rotation $r \in O^+(\mathbb{E})$ that sends $\frac{u}{|u|}$ to $\frac{v}{|v|}$. We have

$$\angle(u, v) = \mathcal{A} \circ \Phi_{\mathbf{b}}(r) \in \mathbb{R}/2\pi\mathbb{Z}$$

For any three distinct points $A, O, B \in \mathbb{E}$ we also define $\angle AOB = \angle(A - O, B - O)$.

From this definition it also follows that the angle θ between two vectors satisfies $\cos\theta = \frac{\langle u,v\rangle}{|u||v|}$. It should be noted that although the cosine of the angle does not depend on the chosen orientation, the sine does and so the inner product alone is NOT enough to determine angles in a two-dimensional vector space up to 2π. In practice what one should do to find the angle between two unit vectors u, v is to extend u to a positively oriented orthonormal basis u, w as above. Then the angle $\angle(u, v)$ is uniquely determined by $\cos\angle(u, v) = \langle u, v\rangle$ and $\sin\angle(u, v) = \langle w, v\rangle$.

One may wonder why we spent so much time discussing angles when the two formulas $\cos\angle(u, v) = \langle u, v\rangle$ and $\sin\angle(u, v) = \langle w, v\rangle$ suffice in practice. The reason is that thinking about angles in terms of rotations makes the addition of angles more intuitive and natural. Indeed it gives an independent proof of the addition formulas for sine and cosine. More to the point it helps us see that $\angle A_1OA_3 = \angle A_1OA_2 + \angle A_2OA_3$ for any vectors $O, A_1, A_2, A_3 \in \mathbb{E}$ satisfying $|A_i - O| \neq 0$ for all i. Assuming $O = 0$ and the A_i are unit vectors for convenience the proof is simply that the rotation taking A_1 to A_2 composed with the rotation taking A_2 to A_3 is the rotation taking A_1 to A_3.

Definition 1.4.2. (Lines, segments, Triangles)
By a line in $\mathbb{E}$ we mean an affine subspace of $\mathbb{E}$ of dimension 1. By a line segment $[AB]$ between two points $A, B \in \mathbb{E}$ we mean the set $\{A + t(B - A) | t \in [0, 1]\}$. For three points $A, M, B \in \mathbb{E}$ the angle $\angle AMB$ is the angle of the unique rotation in $O_M^+(\mathbb{E})$ sending $[MA]$ to $[MB]$. By a triangle we mean a triple of points $A, B, C \in \mathbb{E}$ not contained in a line.

With these definitions in place we can revisit some plane geometry, starting with the angle sum in a triangle.

Lemma 1.4.1. *For any three points $A, B, C \in \mathbb{E}$ we have $\angle ACB + \angle CBA + \angle BAC = \pi$*

Proof. Without loss of generality we choose $C = 0$. Then $\angle ACB = \angle(A, B)$ and $\angle CBA = \angle(-B, A - B)$ and $\angle BAC = \angle(B - A, -A)$. Using the fact that $\angle(u, v) + \angle(v, w) = \angle(u, w)$ and $\angle(-u, -w) = \angle(u, w)$ we find

$$\angle ACB + \angle CBA + \angle BAC = \angle(A, B) + \angle(-B, A - B) + \angle(B - A, -A) =$$

$$\angle(A, B) + \angle(-B, A - B) + \angle(A - B, A) = \angle(A, B) + \angle(-B, A) = \angle(-B, B) = \pi$$

□

Furthermore we have the classic proposition about the angles in the circumference.

Theorem 1.4.1. (Angles in the circumference)
If A, B, C are distinct points in $\mathbb{E}$ and O is the circumcenter (see below) of the triangle A, B, C then

$$\angle AOB = 2\angle ACB$$

Proof. O is the center of the circle through A, C so it is fixed by the (affine) reflection ρ in the perpendicular bisector $\mathcal{H}$ of segment $[A, C]$. This is the affine line of all points of equal distance to both A and C. Since ρ reverses orientation and permutes A and C we have $\angle OCA = \angle CAO$. Lemma 1.4.1 tells us that $\angle OCA + \angle CAO + \angle AOC = 2\angle OCA + \angle AOC = \pi$. In the same way $2\angle BCO + \angle COB = \pi$. Therefore

$$0 = 2\angle OCA + \angle AOC + 2\angle BCO + \angle COB = 2\angle BCA + \angle AOB$$

Since $\angle BCA = -\angle ACB$ we are done. □

Next we turn to the question about the middle of a triangle. What point deserves that name? The centroid, circumcenter and orthocenter all are the middle of a triangle in some sense and they are related in interesting ways.

Define the **centroid** G of triangle $A_0A_1A_2$ to be $G = \frac{A_0+A_1+A_2}{3}$. The segments $[A_iA_j]$ all have midpoints $G_k = A = \frac{A_i+A_j}{2}$ where we take $\{i, j, k\} = \{0, 1, 2\}$. We claim the line three line segments $[G_sA_s]$ all pass through G. Indeed the segment is parametrized by $(1 - t)G_s + tA_s$ and taking $t = \frac{1}{3}$ we recover the formula for G.

A competing notion of the middle of our triangle is known as the **circumcenter** O. It is the center of the unique circle that passes through all vertices A_i. The perpendicular bisector of a line segment $[A, B]$ is the unique line equidistant to both A, B, in other words it is the line orthogonal to $B - A$ that passes through the midpoint $\frac{A+B}{2}$. For a triangle A, B, C the perpendicular bisectors of $[A, B]$ and $[B, C]$ meet in a point O. This point O has the property that it is equidistant from A and B but also equidistant from B and C. That means it is also equidistant from A and C so that the third perpendicular bisector also passes through O and O is the circumcenter.

Finally the **altitudes** of a triangle $\mathcal{A}$ with vertices A_i are the lines $\mathcal{H}_i$ through A_i and orthogonal to the side opposite to A_i. The altitudes meet in a single point called the **orthocenter** $H(\mathcal{A})$ of the triangle. To see why this is the case let us assume without loss of generality that $A_0 + A_1 + A_2 = 0$. Consider the triangle $\mathcal{B}$ with vertices $B_i = -2A_i$. Notice that the line A_iA_j is parallel to B_iB_j since $-2(A_i - A_j) = B_i - B_j$. Also when i, j, k are distinct $\frac{B_j+B_k}{2} = -A_j - A_k = A_i$ so A_i is the midpoint of the segment $[B_jB_k]$. Finally the altitudes of $\mathcal{A}$ are precisely the perpendicular bisectors of the sides of triangle $\mathcal{B}$. The latter meet in the circumcenter of $\mathcal{B}$ so this point is the orthocenter of $\mathcal{A}$.

Looking carefully at what we just proved we find the following theorem of Euler:

Theorem 1.4.2. (Euler line)
The circumcenter, centroid and orthocenter O, G, H of a triangle in $\mathbb{E}$ lie on a line. More precisely $-2(O-G) = H-G$. Also on this line is center N of the circle through the midpoints G_i of the sides of our triangle. Moreover, N is the midpoint of the segment $[H, O]$.

Proof. As in the discussion of the orthocenter we use two triangles $\mathcal{A}, \mathcal{B}$ whose centroid is $G = 0$ and $\mathcal{B} = -2\mathcal{A}$. We denote by G, H, O the centroid, orthocenter and circumcenter of $\mathcal{B}$. By construction $A_i = G_i$, the midpoints of the sides of $\mathcal{B}$. Moreover, we showed the circumcenter $O = H(\mathcal{A})$ the orthocenter of $\mathcal{A}$. By definition this means the orthocenter $H = H(\mathcal{B}) = H(-2\mathcal{A}) = -2O$. We conclude that $O, H, G = 0$ lie on a line and moreover $-2(O - G) = H - G$.

Notice N is the circumcenter $O(\mathcal{A})$ so $-2N = -2O(\mathcal{A}) = O(\mathcal{B}) = O$. Therefore O, G, N, H lie on the same line and $\frac{O+H}{2} = \frac{O-2O}{2} = N$. □

By similar methods much more can be said about the circle through the midpoints of the sides of our triangle. It is the famous nine-point circle.

Theorem 1.4.3. (Nine point circle)
Given triangle $\mathcal{B}$ as above, there is a circle that passes through the following nine special points:

1. *the three midpoints G_i of the sides of* $\mathbf{B}$.
2. *intersections H_i of the altitude with the opposite side of* $\mathbf{B}$.
3. *The midpoints of the segments $M_i = \frac{B_i+H}{2}$.*

Proof. Keeping the previous notation and assuming the centroid G is at the origin define the nine-point circle $\mathcal{C}$ as the unique circle through the $G_i = A_i$, also $\mathcal{C}$ has center $N = \frac{O+H}{2}$. It remains to check that $H_i, M_i \in \mathcal{C}$.

The Euclidean isometry $J : \mathbb{E} \to \mathbb{E}$ defined by $J(x) = 2N - x$ has fixed point N while also mapping M_i to $G_i = A_i$, since $\frac{H}{2} = 2N$. Therefore $|N - M_i| = |J(N) - J(M_i)| = |N - A_i|$ shows $M_i \in \mathcal{C}$.

By construction $G_i, H_i \in [B_j, B_k]$ for i, j, k distinct. Also $H - H_i \perp B_j - B_k$ and $O - G_i \perp B_j - B_k$ by definition of H and O. Set $N_i = \frac{N+G_i}{2}$, then $G_i - N_i$ and $H_i - N_i$ are a scalar multiple of $B_j - B_k$. It follows that $\langle O - G_i, G_i - N_i\rangle = 0 = \langle H - H_i, G_i - N_i\rangle$ so that by bilinearity $\langle N - N_i, G_i - N_i\rangle = 0$. Pythagoras (bilinearity of the inner product) now tells us $|N - G_i|^2 = |N - N_i|^2 + |N_i - G_i|^2$. Likewise $|N - H_i|^2 = |N - N_i|^2 + |N_i - H_i|^2$. The conclusion is that $|N - H_i| = |N - G_i|$ so that $H_i \in \mathcal{C}$ as desired. □

There are in fact many more interesting points on the nine-point circle. For example Feuerbach found there are precisely four circles that are tangent to all three sides of our triangle and the nine-point circle is tangent to all four of those!

Exercises

1. Explain why it is incorrect to speak about *the* Euler line. Give an example of a triangle where there exist several lines through the centers O, G, H.

2. **The end of the Elements**. The final proposition of Euclid's elements deals with the construction of the regular dodecahedron in space (Elements XIII prop. 17.). In this exercise we examine how to do this in our language. Suppose $\dim V = 3$ and consider a regular square with vertices A, B, C, D in an affine plane with $\overrightarrow{AB} = \overrightarrow{DC}$ and a segment $[P, Q]$ with $\phi\overrightarrow{PQ} = \overrightarrow{AB}$

   (a) Prove that we can choose our points such that $|A - B| = 2\phi$ and $|P - Q| = 2 = |P - A| = |P - D| = |Q - B| = |Q - C|$. Here $\phi = \frac{1+\sqrt{5}}{2}$ is the golden number.

   (b) If K, L, M, N, O are the midpoints of segments $[A, B]$, $[C, D]$, $[P, Q]$, $[A, D]$ and $[B, C]$ respectively and the angle between segments $[K, M]$ and $[K, L]$ is κ and the angle between segments $[N, O]$ and $[N, P]$ is ν then prove $\kappa + \nu = \pi$.

   (c) Comment on how your answer from the previous part validates Euclid's construction of the dodecahedron.

3. **Spatial rotations again**. Set $\dim V = 3$ throughout the exercise.

   (a) Explain how an orientation C of V together with a non-zero vector $a \in V$ defines an orientation C' on $a^\perp$.

   (b) For any real number r define an element in $L \in \mathrm{Hom}(V, V)$ by setting $L(a) = a$ and defining the restriction of L to $a^\perp$ be the rotation with angle r in $a^\perp$ and extending linearly. Prove that L is a rotation.

   (c) In the presence of the orientation C of V and non-zero $a \in V$ defines a rotation whose axis is spanned by a and whose angle of rotation is $|a|$. Which vectors define the same rotation in this way?

4. **Law of (co)sines** Consider three distinct points $A_1, A_2, A_3 \in \mathbb{E}$. Considering the indices modulo 3 set $\alpha_i = \angle A_{i-1}A_iA_{i+1}$ and $a_i = |A_{i-1} - A_{i+1}|$. We would like to prove that $\frac{a_i}{\sin \alpha_i}$ is independent of i.

   (a) Define $M_i = \frac{A_{i-1}+A_{i+1}}{2}$ to be the midpoints of the edges of our triangle. Show that $\angle A_{i-1}OM_i = \alpha_i$ where O is the circumcenter of A_1, A_2, A_3.

   (b) Prove that $\sin \alpha_i = \frac{a_i}{2r}$ and hence the law of sines.

   (c) Prove that for any two vectors $u, v \in \mathbb{E}$ we have $\langle u, v\rangle = \frac{|u|^2+|v|^2-|u-v|^2}{2}$.

   (d) Set $A_i = 0$ and prove that $2a_{i-1}a_{i+1} \cos \alpha_i = a_{i-1}^2 + a_{i+1}^2 - a_i^2$.

(e) Argue that this remains true when $A_i \neq 0$ by applying a translation.

5. Consider five points $P_1, \dots P_n$ forming a regular n-gon J and consider the line segments $[P_i, P_{i+2}]$ for any i where we take the index modulo n. Prove that the intersection points $Q_i = [P_i, P_{i+2}] \cap [P_{i+1}, P_{i+3}]$ form another regular n-gon K. What is the ratio between the sides of J and K? Prove that the average of the P_i equals the average of the Q_i and assuming this average is 0 prove that $J = -\lambda K$ for some $\lambda > 0$.

6. Prove that if two triangles S, T have equal angles then there must be a $\lambda > 0$ and an element of the Euclidean group $F \in E(\mathbb{E})$ such that $\lambda S = F(T)$

7. Compute explicitly the angle $\angle(v, -v)$ for non-zero v and also verify that $\angle(-u, -v) = \angle(u, v)$. And what is the relation between $\angle(u, v)$ and $\angle(v, u)$?

8. Prove that for any $r \in O^+(\mathbb{E})$ we have $\angle(r(u), r(v)) = \angle(u, v)$. What can you say about the case where r is a reflection?

1.5 Volume element and Hodge star

We have seen that up to a choice of orientation there is a unique volume element $\nu \in \Lambda^n V$ that may be presented as $\nu = b_{1\dots n} = b_1 \wedge \cdots \wedge b_n$ for any orthonormal basis consistent with the orientation.

Going one step further we define a complementary $(n-k)$-vector to each k-vector in V called the Hodge star. Recall for any sequence I of elements of $\{1 \dots n\}$ and any basis b the element $b_I = \bigwedge_{i \in I} b_i$ where the wedge is in the order of the sequence I. We often assume that I is in increasing order because then the b_I form a basis but the same notation works for any sequence I.

Definition 1.5.1. (Hodge star)
For any orthonormal basis b define $\star_b : \Lambda^k V \to \Lambda^{n-k} V$ by $b_I \wedge \star_b b_I = b_{1\dots n}$ for all k-element sequences $I \subset \{1, \dots, n\}$ and extend linearly.

More concretely this means that $\star_b b_I = \sigma(IJ) b_J$ where J is an $(n-k)$-element sequence in $\{1, \dots, n\}$ such that the concatenation of the sequences (IJ) is a permutation of $\{1 \dots n\}$ with sign $\sigma(IJ)$. For example when $V = \mathbb{R}^3$ with the standard inner product and e is the standard basis then $\star_e e_1 = e_2 \wedge e_3$ and $\star_e e_2 = e_3 \wedge e_1$ also $\star_e 1 = e_{123}$ and $\star_e e_{3,2} = -e_1$.

Just like for the volume element, the Hodge star actually does not depend on the chosen basis, only the orientation matters.

Lemma 1.5.1. (Hodge star)
For any two orthonormal bases b, c we have $\star_b = \pm \star_c$, where the sign is 1 iff the orientations are the same.

Proof. As in the proof of the lemma for the volume element, may assume that $c_i = R_m b_i$ for some reflection given by unit vector $m = \sum_i w_i b_i$. We need to prove that $\star_c c_I = - \star_b c_I$ for any sequence $I \subset \{1, \dots n\}$.

Choose a sequence J complementary to I, and notice $(ij)J$ is complementary to $(ij)I$, where (ij) is the permutation permuting i and j. The right hand side is

$$- \star_b c_I = - \star_b b_I (1 - 2 \sum_{i \in I} w_i^2) + 2 \star_b \sum_{i \in I, j \in J} b_{(ij)I} = -\sigma(IJ) b_J (1 - 2 \sum_{i \in I} w_i^2) + 2 \sum_{i \in I, j \in J} \sigma((ij)I(ij)J) w_i w_j b_{(ij)J}$$

The left hand side is

$$\star_c c_I = \sigma(IJ) c_J = \sigma(IJ) c_J (1 - 2 \sum_{j \in J} w_j^2) - 2\sigma(IJ) \sum_{i \in I, j \in J} w_i w_j b_{(ij)J}$$

The first terms are equal because $1 = |w|^2 = \sum_{i \in I} w_i^2 + \sum_{j \in J} w_j^2$. The second terms are equal because $\sigma(IJ) = -\sigma((ij)I(ij)J)$. □

When the orientation is clear we will simply write $\star$ instead of $\star_b$ for the Hodge star.

Finally we transfer our constructions to dual vectors in V since those will be integrated in the next section. The inner product on V gives in particular a basis-independent isomorphism $V \xrightarrow{\Phi} V^*$ sending v to $\langle v, \cdot \rangle$.

In terms of an orthonormal basis b we have $\Phi(b_i) = b^i$. Identifying V^{**} with V as usual, we get an isomorphism $\Phi^* : \Lambda^k V \to \Lambda^k V^*$. For orthonormal bases it just sends b_I to b^I. This way we obtain a Hodge star $\Lambda^k V^* \xrightarrow{\star} \Lambda^{n-k} V^*$ by $\star b^I = \sigma(IJ) b^J$ much like the above.

Exercises

1. For any linear subspace $F \subset V$ define $V \xrightarrow{s_F} V$ by $s_F(x+y) = x - y$ where $x \in F$ and $y \in F^\perp$. When F is a hyperplane (so the dimension is one less than the whole space) we say the orthogonal isometry is a reflection.

1.6 Simplices and simplicial complexes

In this final section we consider a different aspect of Euclidean or really affine geometry that has to do with multi-dimensional generalizations of triangles called simplices. The tetrahedron is the three-dimensional equivalent of a simplex and such things make sense in any dimension. What kind of things can you build using the simplices as your building blocks? If we ask that the faces of the simplices fit together neatly then we are talking about simplicial complexes. They are quite general so to make sense of them we propose a single integer that will describe some of the essential features: the Euler characteristic.

We start with the more general notion of convexity.

Definition 1.6.1. (Convexity)
$S \subset V$ is called **convex** *if for all $v, w \in S$ we have $tv + (1-t)w \in S$ for any $t \in [0,1]$.*

Inspecting the definition we remark that the intersection of two convex subsets is again convex (Exercise!). This allows us to generate many examples of convex subsets by taking the convex hull. Convexity does not make use of the inner product, it is a notion of affine geometry.

Definition 1.6.2. (Convex hull)
The **convex hull** *$[S]$ of $S \subset V$ is the intersection of all convex subsets of V that contain S.*

For example we already met the the line segment $[A, B] = \{t(b-a) + a | t \in [0,1]\} = [\{A, B\}]$ between two points $A, B \in V$. More generally, the convex hull of a sphere S^{n-1} is the solid ball B contained in it because any point in B is on a straight line between two elements of the sphere. This means that this point must be contained in any convex subset containing the sphere. Triangles can also be viewed by taking the convex hull of three distinct points not on a line. More generally we define k-simplices this way:

Definition 1.6.3. (k-simplex)
By a k-(dimensional)-simplex in V we mean the convex hull $[S]$ of a set S of $k+1$ affine independent vectors in V. For any $T \subset S$ we say that $[T]$ is a face of $[S]$. A facet is a face of dimension $k-1$, for any $s \in S$ there is a corresponding facet $[S - s]$. By definition the empty set is a face of dimension -1.

So a 1-simplex is a line segment, a 2-simplex is a triangle and a 3-simplex is a tetrahedron. A nice way to construct a symmetric k-simplex in V where $k < V$ is by choosing a basis $\mathbf{b}$ of V and taking the convex hull $[b_1, b_2, \ldots b_k]$. For example in $\mathbb{R}^3$ we can make a triangle as $T = [e_1, e_2, e_3]$. The facet opposite to e_2 is $[e_1, e_3]$, a line segment. The reader will have noticed that we tend to abuse notation and drop the extra curly braces, because writing $T = [\{e_1, e_2, e_3\}]$ would be more correct.

Using simplices we can build many other objects and when the faces of the simplices fit together neatly we call such things simplicial complexes.

Definition 1.6.4. (Simplicial complex)
A finite set K of simplices in V is called a simplicial complex if every face of a simplex of K is also in K and the intersection between any two simplices in K is a face of both. The maximum dimension of the simplices of K is called the dimension of K. Define the underlying set to be $|K| = \bigcup_{\sigma \in K} \sigma \subset V$.

A 1-dimensional simplicial complex in V is just a graph embedded in V. Be careful that single k-simplex is not a simplicial complex unless $k = 0$. The union of all faces of a simplex is a simplicial complex. An example of a two-dimensional simplicial complex in $V = \mathbb{R}^3$ is described by

$$K = \{[0, e_1, e_3], [0, e_1], [0, e_3], [e_1, e_3], [0], [e_1], [e_3], [e_1 + 2e_2], [e_3, e_1 + 2e_2], [3e_2 + 3e_1], \emptyset\}$$

Sometimes it is convenient not to list all the simplices in a simplicial complex, just the top-dimensional simplices. So we only list $\sigma \in K$ explicitly if it is not the face of some simplex $\tau \in K$. In the example we would then just write $K = \{[0, e_1, e_3], [e_3, e_1 + 2e_2], [3e_2 + 3e_1]\}$.

With this short-hand we can give any triangular prism $\sigma \times [0,1] \subset \mathbb{R}^{n+1}$ the structure of a simplicial complex when $\sigma = [v_1 \ldots v_k] \subset \mathbb{R}^n \times \{0\}$ is a k-simplex. Set $w_j = v_j + e_{n+1}$ and define the prism as

$$P = \{[v_1, \ldots v_k, w_{k+1}], [v_1, \ldots, v_{k-1}, w_k, w_{k+1}], \ldots, [v_1, v_2, w_3, \ldots, w_{k+1}], [v_1, w_2 \ldots, w_{k+1}]\}$$

Another interesting example is a torus surface in $\mathbb{R}^3$. For any integer $k > 2$ and $0 < r < R$ can parametrize such a surface using a function $\varphi : [0,k)^2 \to \mathbb{R}^3$ given by $(\cos(\frac{2}{k}\pi a)e_1 + \sin(\frac{2}{k}\pi a)e_2)(R + r\cos(\frac{2}{k}\pi b)) + r\sin(\frac{2}{k}\pi b)e_3$. To approximate this shape using a simplicial complex we discretize the variables a, b to only take integer values modulo k. This way we get a simplicial torus T for every value of k, r, R, see also Figure 1.1:

$$T = \bigcup_{a,b=0}^{k-1} \{[\varphi(a,b), \varphi(a+1,b), \varphi(a+1,b+1)], [\varphi(a,b), \varphi(a,b+1), \varphi(a+1,b+1)]\}$$

As before T is a simplicial complex where it is understood that we should also include all the faces of the simplices that are listed.

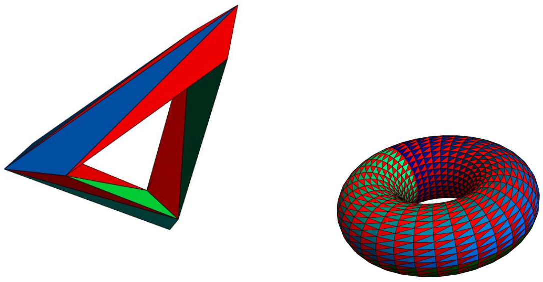

Figure 1.1: Two simplicial complexes that look like a torus. On the left we took $(r, R, k) = (1, 4, 3)$ and on the right $(r, R, k) = (1, 3, 33)$

Simplicial complexes can get quite complicated so it is useful to find common features in them. Clearly a subdividing a simplicial complex should retain some of its properties.

Definition 1.6.5. (Subdivision)
A simplicial complex L is a subdivision of simplicial complex K if $|K| = |L|$ and every simplex in L is contained in a simplex of K.

A famous property preserved under subdivision is the Euler characteristic. It is central to the field of topology and geometry.

Definition 1.6.6. (Euler characteristic)
The Euler characteristic $\chi(K)$ is defined to be $\chi(K) = \sum_{d=0}(-1)^d b_d(K)$, where $b_d(K)$ is the number of d-dimensional simplices in K.

For example if K is the set of all faces of a k-simplex $\sigma \subset V$ then $\chi(K) = 1$. To see why this is true recall that a k-simplex has $k+1$ vertices and any subset of these vertices of size $d+1$ defines a d-dimensional face. Therefore the total number of d-dimensional faces of σ is $\binom{k+1}{d+1}$ and $\sum_{d=0}^{k}(-1)^d\binom{k+1}{d+1} = 1$. For example a tetrahedron aka 3-simplex has Euler characteristic $4 - 6 + 4 - 1 = 1$ because it has 4 vertices, 6 edges, 4 facets and one single solid tetrahedron.

A related but different example is the surface S of a tetrahedron, that is all faces of a tetrahedron except the three-dimensional one, so

$$S = \{[v_0,v_1,v_2],[v_0,v_1,v_3],[v_0,v_2,v_3],[v_1,v_2,v_3],[v_0,v_1],[v_0,v_2],[v_0,v_3],[v_1,v_2],[v_1,v_3],[v_2,v_3],[v_0],[v_1],[v_2],[v_3],\emptyset\}$$

In this case the simplicial complex is two-dimensional and Euler characteristic becomes $\chi(S) = V - E + F = 2$ which is a famous special case of Euler's formula often stated for (surfaces of!) polyhedra.

The fact that the Euler characteristic is invariant under subdivision is not directly clear in general but will follow from Theorem 1.6.1 below. Our technique for making sense of it is to consider collapsing the complex in the following way.

Definition 1.6.7. (Collapse)
We say a nonempty simplex $\tau \in K$ is a free face if there is a unique $\sigma \in K$ that strictly contains it. We say that $L = K - \{\sigma, \tau\}$ is a collapse of K.

For example if our simplicial complex is one-dimensional then collapse means deleting an edge connecting to an endpoint. Another example would be the simplicial complex K consisting of four equilateral triangles in $\mathbb{R}^3$ written as:

$$K = \{[e_i, \frac{e_i+e_j}{2}, \frac{e_i+e_k}{2}], [\frac{e_i+e_j}{2}, \frac{e_j+e_k}{2}, \frac{e_k+e_i}{2}] | \{i,j,k\} = \{1,2,3\}\}$$

where we used the convention that we only write down the top-dimensional simplices, so K is supposed to contain all the faces of the simplices listed. We can take $\tau = [e_1, \frac{e_1+e_2}{2}]$ as a free face since the only simplex that strictly contains it is $\sigma = [e_1, \frac{e_1+e_2}{2}, \frac{e_1+e_3}{2}]$. The collapse is then $K - \{\sigma, \tau\}$.

Notice that the Euler characteristic remains the same under a collapse because we only remove two simplices and the dimension of σ must be one more than that of τ.

It is sometimes helpful to think of a simplicial complex in terms of the graph of inclusions of the sets. This inclusion graph (poset) consists of a vertex for every element of K and a directed edge from A to B if $A \supset B$. In terms of this graph the free face τ and its containing σ look like a source for σ and a vertex for τ that has a single edge to σ and for the rest only edges going into it. The collapse deletes τ, σ and all adjacent edges.

Collapses are not always possible, for example the 1-dimensional simplicial complex $T = \{[e_1, e_2], [e_2, e_3], [e_3, e_1]\}$ in $\mathbb{R}^3$ does not have any free faces.

Using the notion of collapse we can show that Euler characteristic is actually about the space $|K|$ itself and not about how it was built from simplices.

Theorem 1.6.1. *If $|K| = |L|$ then $\chi(K) = \chi(L)$.*

Proof. We can construct a common subdivision of K and L. That way we can assume without loss of generality that L is a subdivision of K.

First we prove the special case where K consists of a single k-simplex and all its faces. If L is a subdivision of K then we want to show that we can collapse L. We do this by scanning L by moving a hyperplane. If we scan a free face we collapse. If we dont get stuck we finish with a single point. If we do get stuck then there has to be a point inside $|K|$ which is not contained in $|L|$. Suppose we sweep past a k-simplex and find no free faces at all. Then there must have been a first such occurrence. But by induction at least one of the previous k-simplices should have collapsed before, contradiction.

Now suppose K is any simplicial complex and L a subdivision. We may simplify L by replacing all the simplices in L that are contained in top-dimensional simplex $\sigma \in K$ by σ. We know this does not change the Euler. Doing so for all top-dimensional simplices of K will turn L into K. □

Of course the converse of the theorem is not true but one could wonder if two complexes with the same Euler characteristic are always connected by a sequence of collapses or their opposites. This is again not true. For example with some effort one can build a Klein bottle inside $\mathbb{R}^4$ and one can also make a torus surface $S^1 \times S^1$. Neither surface has a free face so they cannot be connected by collapses.

The Euler characteristic is very much the beginning, not the end of the study of simplicial complexes. A natural context for the Euler characteristic would the the theory of simplicial homology, a subject in algebraic topology.

Exercises

1. Prove that the intersection between two convex subsets is again convex.

2. Find a simplicial complex K such that $|K| = [0, 1]^3$ and compute its Euler characteristic.

3. Write your first name by constructing a simplicial complex K such that $|K| \subset \mathbb{R}^2$ spells out the letters.

4. Compute the Euler characteristic of the two-dimensional simplicial complex formed by subdividing the faces of a soccer ball into triangles.

5. Look up Bing's house with two rooms and realize it as a simplicial complex. Show that it is not collapsible to a point yet it does have the same Euler characteristic as a point.

6. Sketch a picture of the complexes $K = \{[0, e_1, e_2, e_3, e_4, e_5], [e_1, e_2, e_3, e_4, e_5, e_1 + e_2 + e_3 + e_4 + e_5]\}$ and $L = K \cup \{[e_1 + e_2 + e_3 + e_4 + e_5, 5e_5], [5e_5 - e_1, 5e_5], [-e_1, 5e_5 - e_1], [-e_1, 0]\}$ and compute their Euler characteristics.

7. A convex polyhedron $X \subset V$ is by definition the convex hull of a finite set of points S. X is said to be non-degenerate if V is the only affine subspace containing S. Define the vertex set $\mathrm{Vert}(X)$ of X to be the intersection of all $T \subset V$ such that $[T] = X$. A facet of X is a subset $F \subset X$ of the form $F = X \cap H$ where H is a hyperplane such that $H \cap X^o = \emptyset$. Here X^o is the topological interior of X so it consists of all $x \in X$ that are contained in an open ball within X.

   (a) Verify that an n-simplex $X = [v_0, v_1 \dots v_n] \subset V$ is a non-degenerate polyhedron and that $\mathrm{Vert}(X) = \{v_0, \dots v_n\}$.

   (b) Prove that the facets of X as described above coincide with the facets $[v_1, \dots v_n], [v_0, v_2, \dots v_n], \dots [v_0, v_1 \dots v_{n-1}]$.

   (c) Also prove that the cube $[-1, 1]^n$ is a non-degenerate convex polyhedron in $\mathbb{R}^n$.

(d) By choosing a vertex and connecting it to every other vertex show that any non-degenerate convex polyhedron X may be written as $|K|$ where K consists of a number of n-simplices whose vertices are in $\mathrm{Vert}(X)$.

(e) Compute the Euler characteristic of a convex polyhedron.

8. Let K, L be two 2-dimensional simplicial complexes and suppose $|K| \cap |L| = [p_0, p_1, p_2] = \sigma$ and $K \cap L$ consists of σ and its faces. Show that $F = (K \cup L) - \sigma$ is a simplicial complex and prove $\chi(F) = \chi(K) + \chi(L) - 2$.

9. Find or sketch the construction of simplicial complexes that model a torus and a Klein bottle surface in $\mathbb{R}^4$ and verify that their Euler characteristics are 0.

10. The Möbius strip is one of the most famous geometrical objects. Often it is presented as a smooth surface in $\mathbb{R}^3$ but here we will attempt to build an approximate version of it using a simplicial complex.

    (a) Construct an explicit 2-dimensional simplicial complex M in $\mathbb{R}^3$ such that $|M|$ is a (polygonal) Mobius strip by listing all the 2-simplices in M explicitly. You don't need to list all their faces explicitly, just the triangles.

    (b) Apply as many collapses to M as you can and describe the resulting simplicial complex.

    (c) What is the Euler characteristic of M?

11. Prove that a k-simplex is convex.

Chapter 2

Projective geometry

In this chapter we will investigate projective geometry and how it unifies concepts from affine geometry by adding points at infinity. Although most of our arguments will be valid over an any field we assume V is an n dimensional real vector space unless stated otherwise.

2.1 Projective space

Definition 2.1.1. (Projective space)
For any vector space V define $P(V)$ to be the set of 1-dimensional linear subspaces of V. We define the dimension of $P(V)$ to be $\dim(V)-1$. For a k-dimensional linear subspace $U \subset V$ we say that $P(U) \subset P(V)$ is a $(k-1)$-dimensional projective subspace of $P(V)$.

According to this definition $P(V)$ is empty when $n = \dim(V) = 0$ and a single point when $n = 1$. More importantly, the case $n = 2$ is known as the projective line. The case $n = 3$ is referred to as the projective plane.

One way to visualise projective geometry is in terms of an affine slice in V. By an affine slice we mean an affine hyperplane $f^{-1}(\{1\}) \subset V$ for some $f \in V^*$. All points of V except those "at infinity" can be mapped onto the affine slice as follows:

Lemma 2.1.1. (Affine slice)
For any nonzero $f \in V^*$ there is a bijection $\iota_f : f^{-1}(\{1\}) \to P(V) - P(\ker f)$ sending v to $\mathrm{Span}(v)$. The inverse $\iota_f^{-1}(\mathrm{Span}\ v) = \frac{v}{f(v)}$.

Proof. The map ι_f^{-1} is well-defined because $f(v) \neq 0$. The fact that these maps are inverse follows from $f(\frac{v}{f(v)}) = 1$. □

For example an affine slice for the projective plane $P(\mathbb{R}^3)$ is given by $(\varepsilon^3)^{-1}(\{1\}) = \{(x, y, 1) \in \mathbb{R}^3 | x, y \in \mathbb{R}\}$. The points of $P(\mathbb{R}^3)$ missed by the map ι_{ε^3} are those rays that are parallel to the plane, they are the rays in the x, y-plane $\ker \varepsilon^3$.

Intuitively we think of the origin in V as the observer's eye while the affine subspace $f^{-1}(\{1\})$ is a projection screen onto which most rays reaching the eye will fall. The rays that are missing the screen play the role of points at infinity in perspective drawings onto said affine plane. For example when $V = \mathbb{R}^2$ and $f = \varepsilon^1$ the affine slice of the projective line $P(\mathbb{R}^2)$ is mapped onto the vertical affine line $A = (\varepsilon^1)^{-1}(\{1\})$ by $\iota_{\varepsilon^1}^{-1}$. This map just takes any line through the origin (a point in $P(\mathbb{R}^2)$) and intersects it with A.

For any linear subspace $U \subset V$ we have $P(U) \subset P(V)$. A 1-dimensional projective space is called a projective line. Said differently, every projective line corresponds to a two-dimensional linear subspace of V.

Unlike the parallel lines in affine geometry, any pair of distinct projective lines $P(U), P(W)$ in the projective plane will intersect in a unique point. Indeed since $\dim V = 3$ and $U \neq W$ are two-dimensional linear subspaces we must have $\dim(U \cap W) = 1$ (why?) and so $P(U) \cap P(W) = P(U \cap W)$ is a single point in $P(V)$. In projective space, a pair of projective lines may not intersect at all, just like in affine space. Dually, any two points in $A, B \in P(V)$ determine a projective line denoted AB and defined by $AB = P(A + B)$. Here $A + B$ means the subspace spanned by both A and B.

Definition 2.1.2. (Projective lines, colinear)
For $A \neq B \in P(V)$ we define $AB = P(A + B)$ the line determined by A, B. Points $A, B, C \in P(V)$ are said to be colinear if $AB = AC$.

Occasionally it will be useful to introduce coordinates in $P(V)$ using a basis of V. These are known as homogeneous coordinates.

Definition 2.1.3. (Homogeneous coordinates)
With respect to a basis $\mathbf{b} : \mathbb{R}^n \to V$ *we say* $\mathrm{Span}(v) \in P(V)$ *has homogeneous coordinates* $[h^1 : h^2 : \cdots : h^n]$ *if* $v = \sum_{i=1}^n h^i b_i$.

Any point $p \in P(V)$ has homogeneous coordinates with respect to a basis of V but they are only unique up to a scalar. For example $[1 : 2 : 3]$ and $[-3 : -6 : -9]$ describe the same point $p = \mathrm{Span}(e_1 + 2e_2 + 3e_3) \in P(\mathbb{R}^3)$ in the projective plane. As another example, in the projective line $P(\mathbb{R}^2)$ there are two types of points, those with coordinates $[x : 1]$, a unique one for every value of x and the point at infinity, which has homogeneous coordinate $[1 : 0]$. The points $[x : 1]$ correspond to the affine slice $(\varepsilon^2)^{-1}(1)$.

For studying geometry it is important to consider not just the objects but also the geometric transformations between them. In this case these are known as projective transformations. As in affine geometry the projective transformations will preserve intersections of lines but not shapes and sizes.

Definition 2.1.4. (Projective transformation)
To any injective linear map $L : V \to W$ *we associate a map* $P(L) : P(V) \to P(W)$ *defined by* $P(L)(\mathrm{Span}\ v) = \mathrm{Span}\ L(v)$*, called the associated projective transformation.*

The associated projective transformation just expresses how the linear map sends lines through the origin to other lines through the origin. The assumption of injectivity is to make sure L does not map some line to 0. A simple example would be that $P(\lambda \mathrm{id}_V) = \mathrm{id}_{P(V)}$.

Projective geometry deals with those properties of $P(V)$ that are unchanged under projective transformations. Just like choosing a basis or choosing an orientation we can can use a projective transformation to bring our configurations into a convenient form. For example one may wonder how many points in the projective line can be moved to some desired location by a projective transformation. Since a projective line comes from taking $\dim V = n = 2$ we see that we may send any two lines $\mathrm{Span}(v)$ and $\mathrm{Span}(w)$ to any other two lines, just by finding a linear map L such that $L(v)$ and $L(w)$ are as desired. In fact, we get to choose one more point on the line at will. This motivates the definition of general position:

Definition 2.1.5. (General position)
If $n = \dim V$ *then* $n + 1$ *points of* $P(V)$ *are in general position if ignoring any one of the points we are left with* $\mathrm{Span}(b_1), \mathrm{Span}(b_2), \dots \mathrm{Span}(b_n)$ *for some basis* $b_1, \dots b_n$ *of* V.

For example when $V = \mathbb{R}^2$ the triples of points $\mathrm{Span}(e_1), \mathrm{Span}(e_2), \mathrm{Span}(e_1 + e_2) \in P(\mathbb{R}^2)$ and $\{\mathrm{Span}(e_1 - e_2), \mathrm{Span}(e_1 + 2e_2), \mathrm{Span}(e_1 - 2e_2)\}$ are in general position.

Lemma 2.1.2. *If* $p_1, \dots p_{n+1}$ *are in general position in* $P(V)$ *and* $q_1, \dots q_{n+1}$ *are in general position in* $P(W)$ *then there exists a unique projective transformation* $P(L)$ *such that* $P(L)p_i = q_i$ *for all* $i \leq n + 1$.

Proof. By assumption there are bases $\mathbf{b}, \mathbf{c}$ of V and W such that $p_i = \mathrm{Span} b_i$ and $q_i = \mathrm{Span} c_i$ for all $i \leq n$. Furthermore we may assume that $p_{n+1} = \mathrm{Span}(\sum_{i=1}^n b_i)$ and $q_{n+1} = \mathrm{Span}(\sum_{i=1}^n c_i)$ (Exercise!). It follows that the unique linear transformation $L \in \mathrm{Hom}(V, W)$ defined by $L(b_i) = c_i$ for $i \leq n$ will also map the sum of the b_i to the sum of the q_i so that the associated projective transformation $P(L)$ will be as claimed. □

To illustrate the proof set $f_1 = e_1 - e_2$ and $f_2 = e_1 + 2e_2$. Then $e_1 - 2e_2 = \frac{4}{3} f_1 - \frac{1}{3} f_2$ so take $g_1 = \frac{4}{3} f_1$ and $g_2 = -\frac{1}{3} f_2$. Then define $L \in \mathrm{Hom}(\mathbb{R}^2, \mathbb{R}^2)$ by $Le_i = g_i$. This defines L uniquely and assures us $L(e_1 + e_2) = e_1 - 2e_2$ so that $P(L)$ sends $\mathrm{Span}(e_1), \mathrm{Span}(e_2), \mathrm{Span}(e_1 + e_2)$ to $\mathrm{Span}(e_1 - e_2), \mathrm{Span}(e_1 + 2e_2), \mathrm{Span}(e_1 - 2e_2)$.

Three points on a projective line can be mapped to any other three points but what about four points? We say two quadruple of distinct points (p_1, p_2, p_3, p_4) and (q_1, q_2, q_3, q_4) have the same cross-ratio if there exists a projective transformation T such that $T(p_i) = q_i$ for all i. Using homgeneous coordinates we can describe the cross ratio of a quadruple of points by the number c such that if we find T such that $T(p_1) = [0 : 1]$, $T(p_2) = [1 : 1]$ and $T(p_3) = [1 : 0]$ then $T(p_4) = [c : 1]$.

Theorem 2.1.1. (Desargues)
Imagine six distinct points $A_i, B_i \in P(V)$ *where* $i \in \{0, 1, 2\}$ *such that the three projective lines* $A_i B_i$ *are distinct and meet in* $X \in P(V)$. *Then the three points of intersection* $A_i A_j \cap B_i B_j$ *where* $i \neq j \in \{0, 1, 2\}$ *are colinear.*

Proof. We can find vectors $x, a_i, b_i \in V$ such that $X = \mathrm{Span}(x), A_i = \mathrm{Span}(a_i), B_i = \mathrm{Span}(b_i)$ such that $x = a_i + b_i$ for all $i \in \{0, 1, 2\}$ (why?). For any fixed $i \neq j$ the projective lines $A_i A_j$ and $B_i B_j$ cannot coincide because then the lines $A_i B_i$ and $A_j B_j$ would also coincide. Moreover since $a_j + b_j = x = a_i + b_i$ we have $\mathrm{Span}(a_i - a_j) = \mathrm{Span}(b_i - b_j) = A_i A_j \cap B_i B_j$. Finally, $a_0 - a_1 + a_1 - a_2 + a_2 - a_0 = 0$ so the three points $\mathrm{Span}(a_i - a_j)$ must be on a common projective line (exercise!). □

Another famous theorem in the projective plane is the theorem by Pappus.

Theorem 2.1.2. (Pappus)
In the projective plane $P(V), \dim(V) = 3$ imagine two pairs of colinear triples A_0, A_1, A_2 and B_0, B_1, B_2. The three points of intersection $A_iB_j \cap A_jB_i$, where $i \neq j$, are colinear.

Proof. Exercise! □

Pappus theorem and really any theorem in projective geometry has consequences for affine geometry. Provided we use caution with parallel lines the same statement will hold for affine points and lines by simply taking an affine slice of projective space. For example take two colinear triples of $\bar{A}_0, \bar{A}_1, \bar{A}_2$ and $\bar{B}_0, \bar{B}_1, \bar{B}_2$ in the (affine) vector space W. Provided the three intersections $\bar{C}_{ij} = \bar{A}_i\bar{B}_j \cap \bar{A}_j\bar{B}_i$ exist they are colinear (in the affine sense). The proof of this statement is by mapping W into an affine slice $f^{-1}(\{1\}) \subset V$ using an affine bijection $J : W \to f^{-1}(\{1\})$ for some suitable $f \in V^*$. Then $\iota_f \circ J$ sends the affine points $\bar{A}_i, \bar{B}_i, \bar{C}_{ij}$ to projective points A_i and B_i and C_{ij} that form colinear triples in the projective sense by the projective Pappus theorem.

Exercises

1. Perspective drawing.
   Consider the planes $P, Q \subset \mathbb{R}^3$ defined by $Q = -e_3 + \ker\varepsilon^3$ and $P = e_2 + \ker\varepsilon^2 = (\varepsilon^2)^{-1}(1)$ and the affine lines $L_\pm = -e_3 + \pm e_1 + \ker\varepsilon^1 \cap \ker\varepsilon^3$. We think of $R = L_+ \cup L_- \subset Q$ as the tracks of an infinite railroad R and would like to draw a picture of it on plane P by drawing all the intersections $P \cap r$ where r is a line connecting a point of R to the origin.
   (a) Show that each line r is of the form $\mathrm{Span} v$ with $v \in R$.
   (b) Check that our railroad corresponds to the points $R = \{(\pm 1, y, -1) \in \mathbb{R}^3 | y \in \mathbb{R}\}$.
   (c) Prove that our railroad is mapped onto the following subset of P: $(\pm\frac{1}{y}, 1, -\frac{1}{y}) | y > 0\}$.
   (d) Argue that the point e_2 should correspond to the "point at infinity" where our two lines meet.
   (e) In what sense is the affine line $P \cap \ker(\varepsilon^3)$ the "horizon"?
2. (a) Prove that if $\dim V = 1$ the projective space $P(V)$ is a point.
   (b) Prove that if $\dim V = 2$ then $P(V)$ is in bijection with the union of an affine line and a point.
   (c) Prove that the projective plane is in bijection with a union of an affine plane, an affine line and a point.
   (d) More generally argue that given a basis $\mathbf{b}$ of V we may write
   $$P(V) = \bigcup_{i=1}^{n} P(\bigcap_{j=1}^{i} \ker\beta^j) - P(\bigcap_{j=1}^{i-1} \ker\beta^j)$$
   (e) How do the pieces in the previous formula correspond to affine spaces?
3. Is it true that for nonzero $f \in V^*$ and points $p_0, \dots p_n \subset P(V) - P(\ker f)$ in general position their images $\iota_f^{-1}(p_i)$ are affine independent?
4. Find a bijection between the set of two-dimensional linear subspaces of V and $P(\Lambda^2 V)$.
5. Write down the homogeneous coordinates for the points in $P(\ker f)$ with respect to the basis $b_1 \dots b_n$ of V where $f = \beta^2 - \beta^3$.
6. Consider the shear $S = S_{s,f} : V \to V$ where $f \in V^*$ and $s \in \ker f$ and $S_{s,f}(v) = v - sf(v)$. Check that S is linear and injective and apply the associated projective transformation $P(S)$ to the projective hyperplane $P(\ker f)$, what do you get?
7. Imagine three points $A_0, A_1, A_2 \in P(V)$ with $A_i = \mathrm{Span}(a_i)$. Prove that the A_i are colinear if $a_0 + a_1 + a_2 = 0$. Is the converse also true?
8. Given three distinct points P, Q, R in the projective plane, can we always find a fourth point S such that P, Q, R, S are in general position?
9. Why does it not make sense in projective geometry to talk about the unique line segment connecting points $A, B \in P(V)$?
10. Write down a version of Desargues theorem that is valid for points and lines in affine geometry by choosing an appropriate affine slice for $P(V)$.

11. Show that for any finite number of points in the projective plane there exists a projective line that does not meet any of them.

12. Prove that if ℓ is an affine line passing through $a, b \in f^{-1}(\{1\}) \subset V$ then $\iota_f(\ell)$ is part of a projective line connecting $A = \mathrm{Span}(a)$ and $B = \mathrm{Span}(b)$. Is the converse true? If AB is a projective line in $P(V)$ connecting $A, B \in P$ is $\iota_f^{-1}(AB)$ an affine line connecting $a, b \in V$?

13. Is it true that every projective transformation from V to itself is a bijection?

2.2 Duality

The dual space V^* is a vector space in its own right and as such we may do projective geometry in $P(V^*)$. More interestingly the points of $P(V^*)$ correspond to projective hyperplanes in $P(V)$ and vice versa. For example in the projective plane lines in $P(V)$ correspond to points in $P(V^*)$ and so every statement about the projective plane gives rise to a dual statement by exchanging the roles of points and lines. This is called duality.

Before we begin let us emphasize that even though V^* and V have the same dimension and are isomorphic as vector spaces it is a good habit not to mix the two up lightly as there is no natural (basis independent) isomorphism identifying the two. Identifying V and V^* tacitly chooses an inner product or at least a non-degenerate bilinear form $V \times V^* \to \mathbb{R}$.

In contrast there is a good way to identify the dual of the dual $V^{**} = (V^*)^*$ with the original V. Indeed define $\Psi : V \to V^{**}$ by $\Psi(v)(f) = f(v)$ for all $v \in V, f \in V^*$. Then Ψ is clearly linear and also $\ker \Psi$ must be 0 as a map $f \in V^*$ is 0 if its value on V is 0. Inversely we know a vector $v \in V$ is determined if we know values of all the $f \in V^*$ on v. So we set $z = \Psi^{-1}(\phi) \in V$ to be the unique vector defined by $f(z) = \phi(f)$. In what follows we will sometimes identify V with V^{**} without mentioning Ψ explicitly.

A point in $P(V)$ can be identified with a hyperplane in $P(V^*)$. By a hyperplane in $P(V)$ we mean a subset of the form $P(U)$ where $U \subset V$ is a hyperplane. More generally we introduce

Definition 2.2.1. (Annihilator)
For $U \subset V$ a subspace the **annihilator** $U^o \subset V^*$ *is defined as* $U^o = \{f \in V^* | f(U) = \{0\}\}$.

Of course $f(U)$ just means $\{f(u) | u \in U\}$. When $\dim U = 1$ we have $U \in P(V)$ and $U^o \subset V^*$ and we associate to it $P(U^o) \subset P(V^*)$. More generally we associate to $P(U) \subset P(V)$ the subspace $P(U^o) \subset P(V^*)$. The annihilator changes the direction of inclusions and interchanges the notions of span with intersection of subspaces. With the provision that $V \cong V^{**}$ taking the annihilator twice also brings us back. We list these properties in the following lemma and turn to its geometric consequences after.

Lemma 2.2.1. (Properties of the annihilator)
Suppose $U, W \subset V$ are linear subspaces.

1. *If $U \subset W$ then $W^o \subset U^o$.*

2. $(U \cap W)^o = U^o + W^o$

3. $(U + W)^o = U^o \cap W^o$

4. $\dim U^o + \dim U = \dim V$

5. $(U^o)^o = \Psi(U)$

Proof. For part 4) Choose a basis $b_1, \dots b_k$ of U and extend it to a basis of V by adding vectors $b_{k+1}, \dots b_n$ using the basis extension lemma. If we denote the dual basis by β^i then we have $\beta^i(U) = \{0\}$ if and only if $i > k$. Therefore $\dim U^o = \dim V - \dim U$.

For part 2) Choose a basis $b_1, \dots b_d$ of $U \cap W$ and extend it to a basis of U by adding vectors $b_{d+1}, \dots b_k$. Since the newly added vectors are not in W we can add additional vectors $b_{k+1}, \dots b_{k+s}$ so that $b_1, \dots b_d, b_{k+1}, \dots b_{k+s}$ form a basis of W and the $b_1, \dots b_{k+s}$ form a basis for the span $U + W$. Finally extend to a basis of V by adding some more basis vectors $b_{k+s+1}, \dots b_n$. In terms of the dual basis $\beta^1 \dots \beta^n$ we can describe the annihilators easily: First U^o is spanned by $\beta^{k+1}, \dots \beta^n$ while V^o is spanned by $\beta^{d+1} \dots \beta^k$ and $\beta^{k+s+1}, \dots \beta^n$. Together $U^o + W^o$ are spanned by $\beta^{d+1}, \dots \beta^n$. The same is true for $(U \cap V)^o$ so we established property 2).

The proof of property 3) is similar to property 2).

Property 5) is proven as follows. First $\Psi(U) \subset (U^o)^o$ because for $f \in U^o$ we have $\Psi(u)(f) = 0$ by definition. Since Ψ is an isomorphism property 4) tells us that both sides of 5) have the same dimension so the inclusion shows they must be equal. □

Writing a P in front of each of the statements gives a projective geometry equivalent and we can summarize the situtation as follows.

Theorem 2.2.1. (Projective duality)
There is a bijection $\mathcal{D}$ from the set of projective subspaces of $P(V)$ to those of $P(V^*)$ defined by $\mathcal{D}(P(U)) = P(U^o)$. Then $\mathcal{D}$ reverses inclusions, interchanges span and intersection, turns dimension into codimension and applying it twice gives the identity.

For example in the projective plane $\dim V = 3$ the duality interchanges points with lines. For example a projective line $P(U)$ is associated to a projective point $\mathcal{D}(P(U)) = P(U^o) \subset P(V^*)$. This is indeed a point since the dimensions of U and U^o add up to 3. Under this correspondence two projective lines $P(U), P(W)$ intersecting in point $P(U \cap W)$ become two points $P(U^o)$ and $P(W^o)$ defining a projective line $P(U^o + W^o)$ in $P(V^*)$.

For example if $P(V)$ is a projective plane then a hyperplane is a line so $P(V^*)$ corresponds to the lines in $P(V)$. The space of lines through a point $X \in P(V)$ is precisely the a line X^o in $P(V^*)$. Any two points $X, Y \in P(V)$ define a unique line in $P(V)$ passing through both. From the dual this looks like two lines X^o, Y^o in $P(V^*)$ define a unique point of intersection.

Any result in projective geometry of $P(V)$ can be applied to $P(V^*)$ and then translated back to $P(V)$ to give a dual result.

For example the dual to Desargues theorem in the projective plane is the following:

Theorem 2.2.2. (Dual plane Desargues)
In the projective plane, imagine six distinct projective lines $A_i, B_i \subset P(V)$ where $i \in \{0, 1, 2\}$ such that the three intersection points $A_i \cap B_i$ are distinct and contained in projective line $X \subset P(V)$. Then the three projective lines $(A_i \cap A_j)(B_i \cap B_j)$ where $i \neq j \in \{0, 1, 2\}$ intersect in a single point.

Taken in projective 3-space Desargues also has a dual that looks a little different as in this case the dual to a point is not a line but a projective plane.

Theorem 2.2.3. (Dual 3D Desargues)
In projective three-dimensional space, imagine six distinct projective planes $A_i, B_i \subset P(V)$ where $i \in \{0, 1, 2\}$ such that the three intersection plines $A_i \cap B_i$ are distinct and contained in projective plane $X \subset P(V)$. For $i \neq j \in \{0, 1, 2\}$ denote by C_{ij} the projective plane containing the plines $(A_i \cap A_j)$ and $(B_i \cap B_j)$. The three planes C_{ij} where intersect in a single pline.

The theorem of Pappus theorem likewise has a dual and we leave it as an exercise to find out what it says.

Exercises

1. Define $\mathcal{L}$ to be the set of all affine lines in $\mathbb{R}^2$. In this exercise we aim to construct a bijection between $\mathcal{L}$ and the Mobius strip.
   (a) Show that $\mathcal{L}$ is in bijection with the set of all projective lines in $P(\mathbb{R}^3)$ that have a non-empty intersection with $P(\mathbb{R}^3) - P(\ker \varepsilon^1)$.
   (b) Use projective duality to show that $\mathcal{L}$ is in bijection with the complement of a single point in $P(\mathbb{R}^3)$.
   (c) Show that $P(\mathbb{R}^3)$ is in bijection with the unit sphere in $\mathbb{R}^3$ where we identify all (antipodal points) x and $-x$.
   (d) Using spherical coordinates relate the unit sphere where we leave out the north and south poles and identify the antipodal points to the Mobius strip.
2. Formulate a dual to Pappus theorem in the projective plane.
3. Give an isomorphism of vector spaces $((V^*)^*)^* \to V^*$ that does not depend on a choice of basis.
4. If $F = P(L) : P(V) \to P(V)$ is a projective transformation associated to $L \in \mathrm{Hom}(V, V)$ then show that $\mathcal{D}(F(U)) = P(L^*)(\mathcal{D}(U))$.
5. Suppose V has an inner product. Using the isomorphism $V \to V^*$ sending v to $\langle v, \cdot \rangle$, explain how we may identify U^o with $U^\perp$.
6. Take a basis b_1, b_2, b_3 for V, with dual basis $\beta^1, \beta^2, \beta^3$ of V^*. In what follows we will always refer to points using homogeneous coordinates with respect to these bases.
   (a) Prove that four points in the projective plane $P(V)$ are in general position if and only if no three of them are colinear.
   (b) Explain why any four points in general position $P(V)$ may be mapped to $[1:0:0], [0:1:0], [0:0:1]$ and $[1:1:1]$ using a projective transformation.

(c) Use homogeneous coordinates with respect to the β^i basis to describe the subspaces $\mathcal{D}([1:0:0])$ and $\mathcal{D}([1:1:1])$ of V^*.

(d) Let's say that four projective lines in the projective plane are called in general position if no three of them pass through the same point. Prove that for any four plines in general position there exists a projective transformation that sends these lines to $P(\ker \beta^1), P(\ker \beta^2), P(\ker \beta^3)$ and $P(\ker \beta^1 + \beta^2 + \beta^3)$.

(e) Prove that if projective lines $L_1, \dots L_4$ are in general position and $M_1, \dots M_4$ are also in general position then there is a projective transformation $F : P(V) \to P(V)$ such that $F(L_i) = M_i$.

7. Annihilator.

   (a) Prove that if $U \subset V$ is a linear subspace then U^o is also a linear subspace of V^*.

   (b) If $f \in V^*$ is a non-zero element how is $(\mathrm{Span}(f))^o$ related to $\ker f$?

   (c) When $V = \mathbb{R}^3$, describe the subspace $(\mathrm{Span}(\varepsilon^2 + \varepsilon^3))^o$.

8. In this exercise V has dimension 4 so that we are considering projective 3-space.

   (a) Show that any projective line $P(U) \subset P(V)$ is dual to another projective line in $P(V^*)$.

   (b) Prove that for any three distinct points in $P(V)$ there exists a projective plane $Q \subset P(V)$ that contains all three and that this plane is unique if the points are not in a line.

   (c) Formulate a statement dual to the previous part and prove it. You should start by saying "For any three distinct projective planes in projective space there exists ...."

   (d) If five points in $P(V)$ are in general position, prove that no three of them can belong to the same projective line.

9. In the projective plane $P(V)$ we have five distinct lines L_1, L_2, L_3, L_4, L_5 such that no three intersect in a common point and all their intersections belong to a single line. Formulate the dual configuration in $P(V^*)$ and decide whether this configuration is in fact possible.

2.3 The Grassmannian space of planes through the origin in $\mathbb{R}^4$

Projective space $P(V)$ is the set of all 1-dimensional subsets of V. Said differently the points of $P(V)$ are the lines through the origin in V. Is there also a space $G_2(V)$ whose points correspond to planes through the origin of V? Such spaces are called Grassmannians and their general definition is:

Definition 2.3.1. (Grassmannian)
Define the Grassmannian $G_k(V)$ to be the set of k-dimensional linear subspaces of V. Define the Plücker embedding $\zeta : G_k(V) \to P(\Lambda^k V)$ by

$$\zeta(U) = \mathrm{Span}(\bigwedge_{i=1}^{k} u_i)$$

where $u_1, \dots u_k$ is any basis for $U \in G_k(V)$.

First $G_1(V) = P(V)$ by definition and in the case of $k = 1$ the Plücker map is just the identity. In general the Plücker embedding is well-defined because from the properties of the wedge product we know that if $b_1, \dots b_k$ is another basis of U then $\bigwedge_{i=1}^k b_i = \det(L) \bigwedge_{i=1}^k u_i \neq 0$, where $L : U \to U$ is the unique linear map that satisfies $Lb_i = u_i$.

Next if $\dim(V) = 3$ then we will see that $G_2(V)$ is a projective plane, the Plücker map is a bijection. Indeed if $U \subset V$ is 2-dimensional linear subspace $U = \mathrm{Span}(a, b) \in G_2(V)$ then $\zeta(U) = \mathrm{Span}(a \wedge b) \in P(\Lambda^2 V)$. Of course $\dim \Lambda^2 V = 3$ again and we will see this causes any point in $P(\Lambda^2 V)$ to correspond to a plane $U \subset V$.

The situation when $\dim V = 4$ is very different. Not all points of $P(\Lambda^2 V)$ will correspond to planes in V anymore. For example when $z = e_1 \wedge e_2 + e_3 \wedge e_4 \in \Lambda^2 \mathbb{R}^4$ there is no $U \in G_2(\mathbb{R}^4)$ such that $\mathrm{Span}(z) = \zeta(U)$. Indeed if $U = \mathrm{Span}(b, c)$ then $\zeta(U) = \mathrm{Span}(a)$ with $a = b \wedge c$ so $a \wedge a = 0$. Our vector z does not satisfy $z \wedge z = 0$ so it cannot be a multiple of a. In fact this criterion $a \wedge a = 0$ is sufficient to determine the image of ζ.

Theorem 2.3.1. (Plücker relation)
Assume $\dim V \leq 4$. *The Plücker embedding ζ is a bijection from $G_2(V)$ to* $\{\mathrm{Span}(a) \in P(\Lambda^2 V) | a \wedge a = 0\}$.

Proof. We sometimes say an element $a \in \Lambda^2 V$ is decomposable if $\mathrm{Span}(a)$ is in the image of ζ. In other words if it can be written as a wedge product, not a sum of wedge products. If a is decomposable, say $a = x \wedge y$ with $x, y \in V$ then $a \wedge a = x \wedge y \wedge x \wedge y = 0$. The converse is more interesting and we leave the cases $\dim V \leq 2$ as an exercise to the reader.

If $\dim V = 3$ then choose a basis $\mathbf{b}$ of V and express $a = a^3 b_{12} + a^2 b_{13} + a^1 b_{23}$. The condition $a \wedge a = 0$ is satisfied for any a so we need to show that any a is decomposable. Unless $a = 0 = 0 \wedge 0$ there must be some $a^i \neq 0$, let's say it is a^2 then we can just collect as follows:

$$a = a^3 b_{12} + a^2 b_{13} + a^1 b_{23} = b_1 \wedge (a^3 b_2 + a^2 b_3) + \frac{a^1}{a^2} b_2 \wedge (a^3 b_2 + a^2 b_3) = (b_1 + \frac{a^1}{a^2} b_2) \wedge (a^3 b_2 + a^2 b_3)$$

Finally assume $\dim V = 4$ and write $a = \sum_{i<j} a^{ij} b_{ij}$ for some basis $\mathbf{b}$ of V. Then $a = u \wedge b_4 + a'$ where $\sum_{i=1}^3 a^{i4} b_i \in U = \mathrm{Span}(b_1, b_2, b_3)$ and $a' \in \Lambda^2 U$. The condition $a \wedge a = 0$ means

$$0 = (u \wedge b_4 + a') \wedge (u \wedge b_4 + a') = 2u \wedge b_4 \wedge a' + a' \wedge a'$$

Since a' does not contain b_4 we must have $2u \wedge b_4 \wedge a' = 0$ and $a' \wedge a' = 0$. We already know that $a' = u_1 \wedge u_2$ because a' is in the three-dimensional space U. From the condition $u \wedge 2u \wedge b_4 \wedge a' = 0$ we conclude that $u \wedge u_1 \wedge u_2 = 0$ so either u_1 and u_2 are dependent or $u = \lambda^1 u_1 + \lambda^2 u_2$. In case $u = 0$ we are done because $a = a'$. In the other case we find $a \in \Lambda^2 W$ where $W = \mathrm{Span}(u_1, u_2, b_4)$ to which we may again apply the three-dimensional proof. □

The condition $\dim V \leq 4$ can be relaxed and the same proof will show using induction on $\dim V$ that $G_2(V)$ is characterized by the single quadratic equation $a \wedge a = 0$. The Plücker embeddings of higher Grassmannians ($k \geq 3$) can also be described using quadratic equations but we leave this for a future course in algebraic geometry.

For the remainder of this section we assume $\dim V = 4$ and we investigate some of the geometry of $Q = \zeta(G_2(V)) \subset P(\Lambda^2 V)$. This is a 5-dimensional projective space since $\dim \Lambda^2 V = 6$. Incidentally $G_2(V)$ is also the set of projective lines in three-dimensional projective space $P(V)$. This way we will be able to express some of the geometry of $Q = \zeta(G_2(V))$ in terms of relations between lines in $P(V)$.

As an illustration of the geometry of the Grassmannian we will show that two projective lines ℓ, m in projective 3-space $P(V)$ intersect if and only if the pline $\zeta(\ell)\zeta(m) \subset P(\Lambda^2 V)$ lies entirely in Q. To prove this first suppose $\ell = P(\mathrm{Span} x_1, x_2)$ and $m = P(\mathrm{Span} y_1, y_2)$ for $x_i, y_i \in V$ so $\zeta(\ell) = P(\mathrm{Span} x_1 \wedge x_2)$ and $\zeta(m) = P(\mathrm{Span} y_1 \wedge y_2)$. In terms of this the projective line itself is $\zeta(\ell)\zeta(m) = P(\mathrm{Span} x_1 \wedge x_2 + \mathrm{Span} y_1 \wedge y_2)$. Now if $\mathrm{Span}(z) \in \ell \cap m$ then that means $z = c_1 x_1 + c_2 x_2 = b_1 y_1 + b_2 y_2$ so the four vectors x_1, x_2, y_1, y_2 are linearly dependent. This shows that if $\mathrm{Span}(a) \in \zeta(\ell)\zeta(m)$ then $a = r x_1 \wedge x_2 + s y_1 \wedge y_2$. By assumption $a \wedge a = 2rs x_1 \wedge x_2 \wedge y_1 \wedge y_2 = 0$ because the four vectors are dependent. Conversely, the equation $2rs x_1 \wedge x_2 \wedge y_1 \wedge y_2 = 0$ means the x_1, x_2, y_1, y_2 are dependent so there has to be a $z \in \mathrm{Span}(x_1, x_2) \cap \mathrm{Span}(y_1, y_2)$.

More generally the set of lines in $P(V)$ passing through point $X \in P(V)$ is mapped by ζ to a projective plane contained in Q. This kind of projective plane in Q is called an α-plane.

There is a second type of plane contained in Q known as a β-plane. Suppose $P(W) \subset P(V)$ is a plane. The set of lines in $P(W)$ is mapped by ζ to the set of points $L \in Q$ which lie in a fixed plane contained in Q. One can show that any projective plane contained in Q is either an α-plane or a β-plane.

Exercises

1. Choose a basis $\mathbf{b}$ for V and assume $\dim V = 4$.
   (a) Set $b_{ij} = b_i \wedge b_j$. Are the vectors b_{ij} linearly independent for $i, j \in \{1, 2, 3, 4\}$?
   (b) For $x, y, v, w \in V$ show that $v \wedge w \wedge x \wedge y = x \wedge y \wedge v \wedge w$.
   (c) Write $a = \sum_{i<j} a_{ij} b_{ij}$ and write down a quadratic equation for the coefficients $a_{ij} \in \mathbb{R}$ that is equivalent to $a \wedge a = 0$.
2. Assume $\dim V = 4$ and set $q \in Q = \zeta(G_2(V))$.
   (a) Can q be 0?
   (b) Is q contained in a type α plane in Q? If yes, describe all α planes passing through q in terms of the geometry of $P(V)$.
   (c) Is q contained in a β plane in Q? If yes, describe all β planes through q in terms of the geometry of $P(V)$.
3. Assume $\dim V = 4$.

(a) Give a bijection Z between the set of all projective lines in $P(V)$ and the elements of $G_2(V)$.

(b) Give an example of two projective lines L, M in $P(V)$ that do not intersect also describe their images under Z.

(c) Is the projective line $\zeta(Z(L))\zeta(Z(M))$ contained in $Q = \zeta(G_2(V)) \subset P(\Lambda^2 V)$?

4.

2.4 Affine and projective varieties

In this final section we indicate how projective space is used to study polynomial equations. For simplicity we will only consider $P(\mathbb{R}^n)$ and choose a particular bijection $\iota : \mathbb{R}^n \to P(\mathbb{R}^{n+1}) - P(\ker \varepsilon^{n+1})$ defined by $\iota(x_1, \ldots x_n) = [x_1 : x_2 \cdots : x_n : 1]$.

Definition 2.4.1. (Affine variety)
A polynomial f in variables $x_1, \ldots x_n$ defines a zero-locus $V(f) = \{x \in \mathbb{R}^n | f(x) = 0\}$ called an affine variety

Definition 2.4.2. (Homogeneous polynomial)
A polynomial f in variables $x_1, \ldots x_n$ is said to be **homogeneous** *of degree d if $f(\lambda x_1, \lambda x_2, \ldots \lambda x_n) = \lambda^d f(x_1, x_2, \ldots, x_n)$. A homogeneous polynomial defines a projective variety $V(f) = \{[x_1 : x_2 \cdots : x_n] \in P(\mathbb{R}^n) | f(x_1, x_2, \ldots x_n) = 0\}$.*

Any affine variety $V(f)$ in $\mathbb{R}^n$ can be lifted to a projective variety in $P(\mathbb{R}^{n+1})$ as follows. Start with a polynomial f in variables $x_1, \ldots x_n$. There is a unique minimal degree homogeneous polynomial F in variables $x_1, \ldots x_{n+1}$ such that $F(x_1, \ldots, x_n, 1) = f(x_1, \ldots, x_n)$. To find it, set $d = \deg f(x, x, \ldots x)$ and multiply each monomial with an appropriate power of x_{n+1} to turn the monomial into a homogeneous polynomial of degree d. For example $f(x_1, x_2) = x_1^2 - x_2$ is made homogeneous as $F(x_1, x_2, x_3) = x_1^2 - x_2 x_3$.

In particular we see that $V(f) = \iota^{-1}(V(F) \cap P(\ker \varepsilon^{n+1}))$.

Generally speaking projective varieties have much better properties than their affine counter parts. By adding points at infinity we often obtain a more pleasing or just simpler object. Applying projective transformations to our projective variety can also simplify it even further. To illustrate this point let us consider the parabola $V(f)$ where $f(x_1, x_2) = x_1^2 - x_2$ is made homogeneous as $F(x_1, x_2, x_3) = x_1^2 - x_2 x_3$. The projective counter part $V(F)$ can be brought into a more symmetric form by applying the projective transformation $P(L)$ where $L \in \mathrm{Hom}(\mathbb{R}^3, \mathbb{R}^3)$ is defined by $Le_1 = e_1$, $L(e_2) = e_2 + e_3$ and $L(e_3) = e_2 - e_3$. This amounts to setting $x_1 = y_1$ and $x_2 = y_2 + y_3$ and $x_3 = y_2 - y_3$. Then $0 = F(x_1, x_2, x_3) = y_1^2 + y_2^2 - y_3^2 = G(y_1, y_2, y_3)$. We see that $V(G)$ is a cone when viewed as an affine variety in $\mathbb{R}^3$ and $V(G)$ in $P(\mathbb{R}^3)$ looks like a circle. Or rather $\iota^{-1}(V(G) \cap P(\mathbb{R}^3) - P(\ker \varepsilon^3))$ is the unit circle in $\mathbb{R}^2$.

Much more can be said about projective and affine varieties and the situation often becomes even nicer when one varies the underlying field, from $\mathbb{R}$ to say $\mathbb{C}$ or even $\mathbb{Z}/13\mathbb{Z}$. The reader is invited to verify that all of the projective geometry we discussed in this chapter remains valid over an arbitrary field. Pursuing this topic further brings us into algebraic geometry where for example one thinks about elliptic curves $V(y^2 - x^3 - x + 1)$ projectively as a torus and hence a group.

Exercises

1. (a)

2.

Chapter 3

Riemannian geometry

Our first chapter on Euclidean geometry was about a vector space together with a choice of inner product. In this chapter we follow Riemann's idea of having the inner product be dependent on the point where you are. This allows us to generalize Euclidean geometry to curved spaces. Multivariable calculus is a prerequisite for this part of the course, especially differential forms and Stokes theorem will be used.

3.1 Derivative and differential forms

In this section we recall a few notions from multivariable analysis. Throughout V is a finite dimensional vector space with non-empty open subset $P \subset V$.

For any vector space W consider the function $f : P \to W$. If $p \in P$ the derivative at p is the unique linear map $f'(p) \in \mathrm{Hom}(V, W)$ such that $Err(v) = f(p+v) - f(p) - f'(p)(v)$ is small compared to v. This means $\lim_{v\to 0} \frac{Err(v)}{|v|} = 0$. Intuitively we can often ignore the error and simply write $f(p+v) \approx f(p) + f'(p)(v)$, which is the first order Taylor approximation.

It is important to remember that $f'(p)$ is a linear map, not a matrix. However if $f : \mathbb{R}^n \to \mathbb{R}^m$ and/or we have bases in mind then we often give the matrix of $f'(p)$. This matrix is always the matrix of partial derivatives of f at p. More precisely, the columns of the matrices are the $\partial_i f(p) = f'(p)(e_i)$ for $i = 1 \dots n$.

For a curve $\gamma : (a, b) \to W$ we often use the notation $\dot\gamma(t) = \gamma'(t)(e_1)$ for the tangent vector at point $\gamma(t)$.

Recall that a vector field X on P is a map $X : P \to V$. Likewise a k-form ω on P is a map $\omega : P \to \Lambda^k(V^*)$. If $X_1 \dots X_k : P \to V$ are vector fields then we can evaluate ω on their wedge product to get a real valued function $p \mapsto \omega(p)(\bigwedge_{i=1}^k X_i(p))$. This evaluation is done pointwise. To do so recall that if $\mathbf{b}$ is a basis of V with dual basis vectors $\beta^1, \dots \beta^n \in V^*$ then the vectors b_I where I is an increasing sequence of elements of $\{1, \dots n\}$ and $b_I = \bigwedge_{i\in I} b_i$ form a basis for $\Lambda^k V$. The corresponding dual basis consists of the elements $\beta^I = \bigwedge_{i\in I} \in \Lambda^k V^*$.

The integral of a differential k-form ω on P over a parametrized k-cube $\gamma : [0, 1]^k \to P$ is defined by

$$\int_\gamma \omega = \int_{t\in[0,1]^k} \omega(\gamma(t))(\bigwedge_{i=1}^k \partial_i \gamma(t))$$

where the latter is a k-dimensional Riemann integral.

We also recall the exterior derivative $d\omega$ of a differential k-form ω. It is a $k+1$ form. First when $\omega : P \to \mathbb{R}$ is a function (0-form) then the exterior derivative is defined as $d\omega(p) = f'(p)$. A useful formula in terms of partial derivatives with respect to some basis $\mathbf{b}$ is $df = \sum_{i=1}^n \partial_{b_i} f \beta^i$. If we write $\omega = \sum_I \omega_I \beta^I$ in terms of some basis β of V^* where $\omega^I : P \to \mathbb{R}$ are coefficient functions then $d\omega = \sum_I d\omega^I \wedge \beta^I$.

Stokes theorem states that $\int_\gamma d\omega = \int_{\partial\gamma} \omega$.

Here the boundary $\partial\gamma$ is a linear combination of parametrized k-cubes called a k-chain. We integrate over k-chains by taking the linear combination of the integrals over the cubes involved.

Exercises

1. Compute $\int_\gamma \omega$ where $\gamma : [0, 1]^2 \to \mathbb{R}^3$ is defined by $\gamma(s, t) = se_1 + te_2 + ste_3$ and $\omega(x, y, z) = x\varepsilon^{12} + y\varepsilon^{13}$ defines a 2-form on $\mathbb{R}^3$.

2.

3.

3.2 Riemannian charts

Instead of Euclidean space we propose to work with a more general notion of space that we call a Riemannian chart.

Definition 3.2.1. (Riemannian chart)
A **Riemannian chart** (P, g) *is an open subset* $P \subset V$ *together with a* **Riemannian metric** g*. By a Riemanian metric*[1] g *we mean a choice of inner product* $g(p)$ *for each* $p \in P$ *such that for any two* C^1 *vector fields* X, Y *on* P *the function* $p \mapsto g(p)(X(p), Y(p))$ *is* C^1*.*

If V has an inner product $\langle .,. \rangle$ as in Chapter 1 then we find a simple example of a Riemannian chart by taking $P = V$ and $g(p) = \langle .,. \rangle$ for all $p \in V$. It follows from the chain rule that for C^1-vector fields the function $p \mapsto \langle X(p), Y(p) \rangle$ is a C^1-function. As such the set up of Riemannian charts is a true generalization of the Euclidean geometry in Chapter 1.

A more interesting example of a Riemannian chart is hyperbolic space $(\mathbb{H}^n, g_{hyp})$ where $\mathbb{H}^n = \mathbb{R}^{n-1} \times \mathbb{R}_{>0}$. The Riemannian metric $g_{hyp}(x, y)(v, w) = \frac{1}{y^2}\langle v, w \rangle$, where $\langle .,. \rangle$ is the standard inner product on $\mathbb{R}^n$. More generally whenever $\langle .,. \rangle$ is an inner product on V and $f : P \to (0, \infty)$ a positive function we can define a Riemannian metric g on P by $g(p)(v, w) = f(p)\langle v, w \rangle$. This is still a quite special kind of metrics, called *conformal metrics* not every metric is of this form.

A common way to specify a Riemannian metric is to choose a basis $\mathbf{b}$ for V and then list the coefficient functions of the metric $g_{ij} : P \to \mathbb{R}$ defined by $g_{ij}(p) = g(p)(b_i, b_j)$. Of course the g_{ij} depend on the chosen basis and may not be an appropriate way of describing the metric. Nevertheless the reader should check that if all the g_{ij} are C^1 then so is the function $p \mapsto g(p)(X(p), Y(p))$ for any C^1 vector fields X, Y on P. Our Riemannian metric g gives rise to a norm at every point via $|v|_g(p) = \sqrt{g(p)(v, v)}$ where we take the positive square root. When it is clear from the context that the norm comes from g we will drop the subscript. It is however important to keep the dependence on the point p explicit!

Lengths and angles can still be defined in a Riemannian chart (P, g). Recall the length of a curve is usually obtained by integrating the norm of the velocity vector. As discussed above this norm should be the norm coming from g.

Definition 3.2.2. (Length of a curve)
The **length** *of differentiable curve* $\gamma : [a, b] \to P$ *in the Riemannian chart* (P, G) *is the integral*

$$L(\gamma) = \int_a^b |\dot{\gamma}(t)|_g(\gamma(t))dt$$

The reader should check that the length of the curve is independent of the parametrization of the curve. Let us take a closer look at the hyperbolic plane $\mathbb{H}^2$. For example in the hyperbolic plane the length of the vertical line between $(0, a)$ and $(0, b)$ given by the curve γ defined by $\gamma(t) = (t(b-a) + a)e_2$ with $a < b$ is given by $L(\gamma) = \int_0^1 \frac{b-a}{t(b-a)+a} dt = [\log((b-a)t + a)]_0^1 = \log b - \log a$. As another example consider the wrinkled plane the Riemannian chart $(\mathbb{R}^2, g)$ with $g_{12}(p) = g_{21}(p) = \frac{1}{2}\cos(x)$ and $g_{11}(x, y) = (1 + \frac{1}{2}\sin(x))^2$ and $g_{22}(p) = 1$ and $p = (x, y)$. Then the length of the Euclidean line segment connecting 0 and se_1 is computed as follows. Set $\gamma(t) = te_1$ so $\dot{\gamma(t)} = e_1$ and $L(\gamma) = \int_0^s |\dot{\gamma}|_g(\gamma(t))dt = \int_0^s (1 + \frac{1}{2}\sin(t))dt = [t - \frac{1}{2}\cos t]_0^s = s + \frac{1-\cos s}{2}$.

We can also define the angle between two curves that intersect in a point. For this we do need an orientation at every point.

Definition 3.2.3. (Orientation)
An **orientation** *on* P *is a choice of orientation* $C(p)$ *of* V *for every* $p \in P$ *that is constant on the connected components of* P*.*

Often we just pick a single orientation J on V and set the orientation C on P to be just $C(p) = J$ for all $p \in P$.

Definition 3.2.4. (Angle between intersecting curves)
Suppose $\beta, \gamma : [a, b] \to P$ *are curves in Riemannian chart* (P, g) *and choose an orientation* C *on* P*.*
If $\beta(q) = \gamma(q) = p \in P$ *and* $\dot{\beta}(q), \dot{\gamma}(q) \neq 0$ *then we define the* **angle** *between* β, γ *at* p *to be the angle between* $\dot{\beta}(q), \dot{\gamma}(q)$ *in the inner product space* $(V, g(p))$ *with orientation* $C(p)$ *in the sense of Definition 1.4.1.*

Any conformal metric such as the hyperbolic metric has the property that the angles are just the Euclidean angles. In our wrinkled plane example the angle between curves $\beta(t) = te_1$ and $\gamma(t) = te_2$ at point $p = 0$ is more interesting. We fix the orientation by stating that e_1, e_2 is a positively oriented basis. $\dot{\beta}(0) = e_1$ and $\dot{\gamma}(0) = e_2$ so we just need to know the angle between e_1 and e_2 with respect to the metric $g(0)$. We have

[1] Be careful the name 'metric' clashes with the usage of 'metric' as in metric spaces!

$g(0)(e_1, e_1) = 1 = g(0)(e_2, e_2)$ but $g(0)(e_1, e_2) = \frac{1}{2}$ so e_1 and e_2 are unit vectors but they are not orthogonal. We apply Gram-Schmidt and set $b_1 = e_1$ and $b_2 = N(e_2 - g(0)(b_1, e_2)b_1)$ where the N means to divide by the norm. So $b_2 = \frac{2}{\sqrt{3}}(e_2 - \frac{1}{2}e_1)$. This basis is still positive. The cosine of the angle is $g(0)(b_1, e_2) = \frac{1}{2}$ and the sine is $g(0)(b_2, e_2) = \frac{2}{\sqrt{3}}(1 - \frac{1}{4}) = \frac{\sqrt{3}}{2}$ so the angle must be $\pi/3$.

Definition 3.2.5. (Orthonormal frame, volume n-form)
An **othonormal frame** *on P is a sequence of C^2-vector fields $X_1, \dots X_n : P \to V$ such that for all $p \in P$ the vectors $X_1(p), \dots X_n(p)$ form an orthonormal basis for V with respect to $g(p)$. The dual 1-forms $\xi^1 \dots \xi^n \in \Omega^1_2(P)$ are defined by $\xi^i(p)(X_j(p)) = \delta^i_j$. Finally the volume n-form $\nu \in \Omega^n_2(P)$ is defined as $\nu_g(p) = \bigwedge_{i=1}^n \xi^i(p)$.*

In hyperbolic space an orthonormal frame is given by $X_i(x, y) = ye_i$. The corresponding dual 1-forms are $\xi^i(x, y) = \frac{1}{y}\varepsilon^i$ and the volume n-form is $\nu = \frac{1}{y^n}\varepsilon^1 \wedge \varepsilon^2 \cdots \wedge \varepsilon^n$.

Definition 3.2.6. (Volume)
The volume of a parametrized n-cube $\gamma : [0, 1]^n \to P$ in (P, g) is the integral $Vol(\gamma) = \int_\gamma \nu_g$.

For example the volume of the parametrized 2-cube in the hyperbolic plane $\gamma : [0, 1]^2 \to \mathbb{H}^2$ given by $\gamma(s, t) = (s, t+h)$ is $\int_\gamma \nu = \int_{(s,t)\in[0,1]^2} \nu(\gamma(s, t))(\partial_1\gamma(s, t) \wedge \partial_2\gamma(s, t)) = \int_{(s,t)\in[0,1]^2} \frac{1}{(t+h)^2}\varepsilon^{12}(e_1 \wedge e_2) = \int_{(s,t)\in[0,1]^2} \frac{1}{(t+h)^2} = -\frac{1}{1+h} + \frac{1}{h}$.

Metrics can also be transferred by pull-back as follows.

Definition 3.2.7. (Pull-back metric)
Suppose V, W are vector spaces and $\varphi : P \to Q$ is a C^2 map between open subsets $P \subset V$ and $Q \subset W$. If g is a Riemannian metric on Q and $\varphi'(p)$ is injective for all $p \in P$, we may define a metric φ^*g on P by $(\varphi^*g)(p)(v, w) = g(\varphi(p))(\varphi'(p)v, \varphi'(p)w)$.

The derivative of φ is required to be injective in order to ensure non-degeneracy of the pulled back inner product. Indeed, the reader should try to prove that the pull-back metric is indeed a Riemannian metric. For example symmetry of $h = \varphi^*g$ follows from symmetry of g because $h(p)(a, b) = g(\varphi(p))(\varphi'(p)(a), \varphi'(p)(b)) = g(\varphi(p))(\varphi'(p)(b), \varphi'(p)(a)) = h(p)(b, a)$. The other properties of the inner product are checked similarly (Exercise!)

For example we may take the sphere and geographic coordinates $G = (0, \pi) \times (-\pi, \pi)$ and $G \ni (\mu, \lambda) \overset{\varphi}{\mapsto} (\cos\lambda \sin\mu, \sin\lambda \sin\mu, \cos\mu) \in \mathbb{R}^3$. Here μ is the latitude coordinate and λ the longitude, for example Groningen is the point $\varphi(53.217\frac{\pi}{180}, 6.567\frac{\pi}{180})$. Explicitly the inner product φ^*g_{Eucl} is given by calculating it at every point for the basis vectors e_1, e_2. Since the matrix for $\varphi'(p)$ is $\begin{pmatrix} \cos\lambda\cos\mu & -\sin\lambda\sin\mu \\ \sin\lambda\cos\mu & \cos\lambda\sin\mu \\ -\sin\mu & 0 \end{pmatrix}$ we get $\varphi^*g_{Eucl}(p)(e_1, e_1) = 1$, $\varphi^*g_{Eucl}(p)(e_1, e_2) = 0$, $\varphi^*g_{Eucl}(p)(e_2, e_2) = \sin^2\mu$.

More generally we can attempt to describe any level set $S = f^{-1}(\{0\})$ of some C^1-function $f : \mathbb{R}^{m+n} \to \mathbb{R}^m$ using a Riemannian chart. From the implicit function theorem we know that if the tangent space T_sS is the graph of a linear function then, close to $s \in S$ we may write S as the graph of a $h : X \to Y$ function too. Then $\varphi : \mathbb{R}^n \to \mathbb{R}^{n+m}$ given by $\varphi(p) = (p, h(p))$ is C^1 and also $\varphi'(p)$ is injective so we can use it to pull-back the Euclidean metric on $\mathbb{R}^{n+m}$ and find a Riemannian chart (X, φ^*g_E).

Given a geometric object $Q \subset \mathbb{R}^n$ in Euclidean space there are two ways to study it: **intrinsically** or **extrinsically**. The intrinsic way of studying Q would be to find a differentiable map $P \xrightarrow{\varphi} Q$ if injective derivative and turn P into a Riemannian chart (P, g) as above by pulling back the Euclidean metric: $g = \varphi^*g_{Eucl}$. Properties of Q that can be described in this way are called **intrinsic** properties. The length of a curve $\gamma : [0, 1] \to Q$ is an example of an intrinsic property.

The other way is to work with Q from the 'outside' or **extrinsically** by studying how it sits in $\mathbb{R}^n$ using the tools from Euclidean geometry and differential calculus. A typical extrinsic property of Q is the Euclidean distance between two points in Q.

Both the intrinsic and extrinsic are important but in modern geometry there is an emphasis on the intrinsic features. Often working intrinsically is more efficient in that one needs less coordinates and is less distracted by 'irrelevant' extrinsic features. Also sometimes our space Q is not presented to us as subset of some Euclidean space but still has a Riemannian metric. In that case the intrinsic approach is the only option. For example when Q is the universe or when $Q = O(E)$ the orthogonal group.

Definition 3.2.8. (Isometry)
A diffeomorphism is a differentiable bijection with differentiable inverse. A (Riemannian) isometry is a diffeomorphism that preserves the metric in the following sense. $(P, g) \xrightarrow{\varphi} (Q, h)$ is an isometry if $g = \varphi^*h$.

Since all geometric properties derive from the metric, isometries can be thought of as those transformations that preserve shape. As the name suggests the Euclidean linear isometries $O(E)$ from a Euclidean vector space to itself are examples of the above Riemannian notion of isometry. In this case we have $(P, g) = (Q, h) = (\mathbb{R}^n, g_E)$, the standard inner product in every point.

Other examples of isometries arise when we describe the same object in Euclidean space using two different Riemannian charts. For example let us make a completely different chart describing (part of) the sphere in $\mathbb{R}^3$. The stereographic map $\sigma : \mathbb{R}^2 \to S^2 \subset \mathbb{R}^3$ is defined as $\sigma(p) = \frac{(2p,|p|^2-1)}{|p|^2+1}$ where $p = (p_1, p_2)$. It has inverse $\sigma^{-1}(x, z) = \frac{x}{1-z}$ where $x = (x_1, x_2) \in \mathbb{R}^2$ defined on all of S^2 exept the north pole where $z = 1$.

The stereographic map σ has injective derivative at every point and is a C^2 function. Therefore we can use it to turn $P = \mathbb{R}^2$ into a Riemannian chart (P, g) with pull-back metric $g = \sigma^* g_E$. Recall we also had the geographic Riemannian chart $(G, \varphi^* g_E)$ of the S^2. The map $\sigma^{-1} \circ \varphi$ when restricted to the correct domain is an example of an isometry between Riemannian charts.

Another simple example is an isometry of $t : \mathbb{H}^2 \to \mathbb{H}^2$ defined by $t(x, y) = (x + 1, y)$. Since $t'(p)$ is the identity at any point p we get $t^* g_{hyp} = g_{hyp}$. There are many more isometries of $\mathbb{H}^2$ and we will get back to them later.

A more intricate example of an isometry is the following. Introduce the (Poincaré) disk model $\mathbb{D} = (\{u \in \mathbb{C} : |u| < 1\})$ with metric $g(u) = \left(\frac{2}{1-|u|^2}\right)^2 g_{Eucl}(u)$. Here as usual we identify $\mathbb{R}^2$ and $\mathbb{C}$. Now we claim that $\mathbb{D} \xrightarrow{\phi} \mathbb{H}$ given by $\phi(u) = \frac{u+i}{iu+1}$ is an isometry. Here we also identified the upper half plane $\mathbb{H}$ with a subset of $\mathbb{C}$.

To verify this we compute using complex numbers as much as possible to avoid lengthly expressions. First we use the fact that for a complex differentiable function $f : \mathbb{C} \to \mathbb{C}$ we may identify the derivative (a linear transformation in $L(\mathbb{R}^2, \mathbb{R}^2)$) with multiplication by $f'(z)$. Also identifying a vector v by $a + ib$ the Euclidean inner product becomes $\langle v, w\rangle = \mathrm{Re} v\bar{w}$. The metric on $\mathbb{H}$ is then written as $g_{hyp}(z)(v, w) = \frac{\mathrm{Re} v\bar{w}}{(\mathrm{Im}(z))^2}$. The pull back of this metric along ϕ becomes $\phi^* g_{hyp}(u)(v, w) = \mathrm{Re}\frac{\phi'(u)v\overline{\phi'(u)w}}{(\mathrm{Im}(\phi(u)))^2} = \frac{|\phi'(u)|^2}{(\mathrm{Im}(\phi(u)))^2}\mathrm{Re}(v\bar{w}) = \frac{|\phi'(u)|^2}{(\mathrm{Im}(\phi(u)))^2} g_{Eucl}(u)$. Finally $|\phi'(u)|^2 = \frac{4}{|iu+1|^4}$ and $\mathrm{Im}\phi(u) = \frac{1-|u|^2}{|iu+1|^2}$ finishing the computation.

A similar computation shows that isometries of the hyperbolic plane $\mathbb{H}$ are given by the linear fractional tranformations $z \mapsto \frac{az+b}{cz+d}$ where $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$. In fact these are all orientation preserving isometries but we will not show this here.

Exercises

1. Suppose $g_{ij} : P \to \mathbb{R}$ are C^1-functions and $\mathbf{b}$ is a basis for V. Prove that if for any $p \in P$ the $n \times n$ matrix $(g_{ij}(p))$ is positive definite then g defines a Riemannian metric on P.

2. Compute the length of the curve $\gamma : [0, 1] \to \mathbb{H}$ with $\gamma(t) = A(1 - t) + Bt$ connecting $A = e_n$ and $B = \lambda e_1 + e_n$ in hyperbolic n-space $\mathbb{H}^n$. How would you shorten your trip from A to B? Detour upwards in the e_n direction or downwards in the $-e_n$ direction? Can you find a path $\alpha : [0, 1] \to \mathbb{H}$ between A, B that is shorter than γ?

3. Set $p = (x, y) \in P = \mathbb{R}^2 = V$ and extend $g(p)(e_1, e_1) = e^x$ and $g(p)(e_2, e_2) = e^{2y+x}$ to a bilinear function $g(p) : \mathbb{R}^2 \to \mathbb{R}$. Does this define an Riemannian metric on $\mathbb{R}^2$? What is $g_{11}(p)$?

4. Verify that in the sphere example the volume 2-form in μ, λ coordinates must be $(\sin\mu)\varepsilon^1 \wedge \varepsilon^2$ because a positive orthonormal basis is $e_1, (\sin\mu)^{-1}e_2$. Recall $\varepsilon^i \in (\mathbb{R}^2)^*$ are the dual basis elements. Volume is now defined by integrating the volume 2-form. Can you compute the volume of a spherical triangle?

5. Hyperbolic plane. In this exercise we identify $z = x+iy \in \mathbb{C}$ with $(x, y) \in \mathbb{R}^2$ in the standard way. Likewise a complex differentiable function $f : \mathbb{C} \to \mathbb{C}$ is identified with a differentiable function $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ and multiplication by $f'(z)$ provides a linear map from $\mathbb{R}^2$ to itself that coincides with the derivative $\phi'(x, y)$. Next, the Euclidean inner product becomes $\langle v, w\rangle = \mathrm{Re}(v\bar{w})$.

   (a) For $a, b, c, d \in \mathbb{R}$ with $ad - bc = 1$ set $f(z) = \frac{az+b}{cz+d}$. Check that $f'(z) = (cz + d)^{-2}$ and $\mathrm{Im} f(z) = \mathrm{Im}(z)|cz + d|^{-2}$.

   (b) Identifying the set $\{z \in \mathbb{C} : \mathrm{Im} z > 0\}$ with the hyperbolic plane $\mathbb{H}$, show that the map f sends the hyperbolic plane to itself and that the hyperbolic metric g is written as $g(z)(v, w) = \frac{\mathrm{Re}(v\bar{w})}{(\mathrm{Im} z)^2}$.

   (c) Prove that the maps f (restricted to $\mathbb{H}$) are isometries in the sense that $g(f(z))(f'(z)v, f'(z)w) = g(z)(v, w)$.

6. For any $r \in (0, 1)$ consider the length of the curve $\gamma : [0, 2\pi] \to \mathbb{R} \times [0, \infty)$ given by $\gamma(t) = r(\cos(t)e_1 + \sin(t)e_2) + e_2$ with respect to the Euclidean metric and also with respect to the hyperbolic metric. Which is bigger? Is this true for all values of r or is there some point where both lengths agree?

7. Imagine a complex differentiable function $f : \mathbb{C} \to \mathbb{C}$ and construct its graph $G = \{(z, f(z))|z \in \mathbb{C}\} \subset \mathbb{C}^2$. We view $\mathbb{C}^2$ as a Riemannian chart with Euclidean metric g_E. More precisely, $g_E(p)(v, w) = \Re(v_1\bar{w}_1) + \Re(v_2\bar{w}_2)$ where $v = (v_1, v_2) \in \mathbb{C}^2$.
   (a) Prove that $\phi : \mathbb{C} \to \mathbb{C}^2$ given by $\phi(z) = (z, f(z))$ has injective derivative $\phi'(z)$ for any $z \in \mathbb{C}$.
   (b) Compute the pull-back metric $\phi^* g_E$.
8. Consider $\varphi : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $\varphi(a, b) = (a, b, \sin(a + b))$.
   (a) Prove that $\varphi'(p)$ is injective for all $p \in \mathbb{R}^2$.
   (b) Compute $g_{ij}(p) = g(p)(e_i, e_j)$ where $g = \varphi^* g_E$ is the pull-back of the Euclidean metric on $\mathbb{R}^3$.
   (c) Find an orthonormal frame $X_1, X_2 : \mathbb{R}^2 \to \mathbb{R}^2$ with respect to the Riemannian metric g.
   (d) Write down the volume 2-form on corresponding to your frame X_1, X_2.
   (e) Compute the area with respect to g of the parametrized 2-cube $\gamma : [0, 1]^2 \to \mathbb{R}^2$ given by $\gamma(p) = p$.
9. (a) Pull back the standard Euclidean (Riemannian) metric on $\mathbb{R}^3$ using $\varphi : \mathbb{R}^2 \to \mathbb{R}^3$ given by $\varphi(x, y) = (x, y, x^2 + y^2)$.
   (b) Compute the circumference of the Euclidean circle with radius $r > 0$ in the Riemannian chart $(\mathbb{R}^2, \varphi^* g_E)$.
   (c) Assuming that the straight Euclidean line connecting 0 to (x, y) is the shortest distance between these points (this seems plausible for symmetry reasons) figure out what the circle of radius $r > 0$ and center 0 looks like when we use the metric g to measure distance. What is its circumference?
10. Prove that if g is a Riemannian metric on Q and φ is as in Definition 3.2.7 then $\varphi^* g$ is also a Riemannian metric.
11. Show that the composition of two Riemannian isometries is again an isometry.

3.3 Moving frames in Euclidean space

In this section we will work in Euclidean space (P, g) so $P \subset V$ and V an inner product space and $g(p)(v, w) = \langle v, w\rangle$ for all $p \in P, v, w \in V$. Already in this relatively simple setting an orthonormal frame $X_1, \ldots X_n : P \to V$ is an interesting geometric object. Recall the dual frame (coframe) is denoted by $\xi^1, \ldots \xi^n : P \to V^*$.

How does the frame vary as we move around in space? As it moves around we can describe its movement in terms of itself. This notion is captured by the connection 1-forms. Later on we will be thinking of the frame as consisting of a few vectors spanning the tangent plane to a surface and the remaining ones making up the normal vector(s). The movement of the frame thus captures the shape of the surface.

Definition 3.3.1. (connection 1-forms)
For $1 \le i, j \le n$ define the connection 1-forms $\omega_i^j : P \to V^*$ by $\omega_i^j(p) = \xi^j(p) \circ X_i'(p)$.

Another way to express the connection 1-forms is by saying $X_i'(p)(v) = \sum_j \omega_i^j(p)(v) X_j(p)$. Taking $v = X_k(p)$ we see that the coefficients of $\omega_i^j(p)$ with respect to the basis ξ^k are the matrix elements for the matrix of X_i' with respect to the X_k basis.

As a simple example is $P = \mathbb{R}^2 - \{(0, 0)\}$ and set $p = (a, b)$ and $X_1(p) = \frac{p}{|p|}$ and $X_2(p) = \frac{(-b,a)}{|p|}$. We have $\xi^1(p) = \langle X_1(p), e_1\rangle\varepsilon^1 + \langle X_1(p), e_2\rangle\varepsilon^2 = \frac{a\varepsilon^1+b\varepsilon^2}{|p|}$ and likewise $\xi^2(p) = \frac{-b\varepsilon^1+a\varepsilon^2}{|p|}$. The derivatives $X_1'(p)$ and $X_2'(p)$ have the following matrices when we compute them with respect to the standard bases:

$$\frac{1}{|p|^3}\begin{pmatrix} b^2 & -ab \\ -ab & a^2 \end{pmatrix} \qquad \frac{1}{|p|^3}\begin{pmatrix} ab & -a^2 \\ b^2 & -ab \end{pmatrix}$$

It follows that $X_1'(p)(X_1(p)) = 0 = X_2'(p)(X_1(p))$ while $X_1'(p)(X_2(p)) = \frac{X_2(p)}{|p|}$ and $X_2'(p)(X_2(p)) = -\frac{X_1(p)}{|p|}$. Applying ξ^j to these equations tells us the value of ω_i^j on the X basis. Since $\omega_i^j = \omega_i^j(X_1)\xi^1 + \omega_i^j(X_2)\xi^2$ we conclude $\omega_i^i = 0$ while $\omega_2^1 = -\omega_1^2$ and $\omega_1^2 = \frac{\xi^2}{|p|}$.

The relation $\omega_i^j = -\omega_j^i$ is in fact always valid and follows from $\langle X_i, X_j\rangle = \delta_{ij}$. To see why recall that if we define $G : V \times V \to \mathbb{R}$ by $G(v, w) = \langle v, w\rangle$ then $G'(p, q)(v, w) = \langle p, w\rangle + \langle v, q\rangle$. Introducing $H : V \to \mathbb{R}$ by $H(p) = \langle X_i(p), X_j(p)\rangle$ we have $H = G \circ (X_i, X_j)$ so by the chain rule $H'(p)(v) = G'(X_i(p), X_j(p))(X_i'(p)v, X_j'(p)v) = \langle X_i(p), X_j'(p)v\rangle + \langle X_i'(p)v, X_j(p)\rangle$. Since $\xi^i(p)(v) = \langle X_i(p), v\rangle$ and $H'(p) = 0$ the relation follows.

Another way to look at a moving frame is to choose a reference basis $b_1, \ldots b_n$ of V. Then any orthonormal frame $X_1, \ldots X_n$ on P can be written as $X_i(p) = L(p)b_i$ for some function $L : P \to O(V) \subset \mathrm{Hom}(V, V)$. In our example on $\mathbb{R}^2 - \{(0,0)\}$ we could take our reference basis to be $b_i = e_i$ the standard basis and then $L(p) = R_{m(p)} \circ R_{e_2}$ where $m(p) = \frac{p - e_1|p|}{|p - e_1|p||}$.

The connection forms satisfy many other relations known as the structure equations:

Lemma 3.3.1. (Structure equations)
The connection 1*-forms of the orthonormal frame* $X_1, \dots X_n$ *of* V *satisfy:*

1. $\omega_i^j = -\omega_j^i$
2. $d\xi^i = \sum_{k=1}^n \xi^k \wedge \omega_k^i$
3. $d\omega_i^j = \sum_{k=1}^n \omega_i^k \wedge \omega_k^j$

Proof. We already proved part 1) and for part 2) the idea is to carry out the computations explicitly in terms of a basis $e_1 \dots e_n$ of V. Define functions $L_i^j : P \to \mathbb{R}$ by $L_i^j(p) = \langle X_i(p), e_j \rangle$. Since $X_i = \sum_j L_i^j e_j$ we have $\delta_s^r = \langle X_r, X_s \rangle = \sum_j \langle X_r, L_s^j e_j \rangle = \sum_j L_r^j L_s^j$. We also have $\xi^i = \sum_j L_i^j \varepsilon^j$ (exercise!) and taking the exterior derivative of this equation yields

$$d\xi^i(p) = \sum_j dL_i^j \wedge \varepsilon^j \tag{3.1}$$

By definition $X_i'(p)(v) = \sum_j (L_i^j)'(p)(v) e_j = \sum_k \omega_i^k(p)(v) X_k(p) = \sum_{k,j} \omega_i^k(p)(v) L_k^j(p) e_j$ and so $dL_i^j(p)(v) = (L_i^j)'(p)(v) = \sum_k \omega_i^k(p)(v) L_k^j(p)$. Combining (3.1) and the previous proves part 2): $d\xi^i(p)(v) = \sum_j dL_i^j(p)(v) \wedge \varepsilon^j = \sum_{j,k} \omega_i^k(p)(v) L_k^j(p) \wedge \varepsilon^j = \sum_{j,k} \omega_i^k(p)(v) \wedge L_k^j(p)\varepsilon^j = \sum_k \omega_i^k(p)(v) \wedge \xi^k$. Finally Part 3) is proven by taking the exterior derivative of $dL_i^j = \sum_k \omega_i^k L_k^j$ and using $\delta_s^r = \sum_j L_r^j L_s^j$: $0 = ddL_i^j = \sum_k (d\omega_i^k) L_k^j - \omega_i^k \wedge dL_k^j$. So $\sum_k (d\omega_i^k) L_k^j = \sum_{k,s} \omega_i^k \wedge \omega_k^s L_s^j$ arriving at $d\omega_i^r = \sum_{k,j} (d\omega_i^k) L_k^j L_r^j = \sum_{k,s,j} \omega_i^k \wedge \omega_k^s L_s^j L_r^j = \sum_k \omega_i^k \wedge \omega_k^r$ ☐

In our basic example we may verifty parts 2) and 3) of the structure equations by explicitly computing the exterior derivatives of $\xi^1(p) = \frac{a\varepsilon^1 + b\varepsilon^2}{|p|} = f\varepsilon^1 + g\varepsilon^2$. By definition $d\xi^1(p) = df(p) \wedge \varepsilon^1 + dg \wedge \varepsilon^2 = -\frac{ab}{|p|^3}\varepsilon^2 \wedge \varepsilon^1 - \frac{ab}{|p|^3}\varepsilon^1 \wedge \varepsilon^2 = 0$ which is consistent because we already saw that $\omega_2^1 = -\frac{\xi^2}{|p|}$. In the same way $d\xi^2(p) = -\frac{a^2}{|p|^3}\varepsilon^2 \wedge \varepsilon^1 + \frac{b^2}{|p|^3}\varepsilon^1 \wedge \varepsilon^2 = \frac{\varepsilon^1 \wedge \varepsilon^2}{|p|} = \frac{\xi^1 \wedge \xi^2}{|p|}$. This is consistent with $\omega_1^2 = \frac{\xi^2}{|p|}$. For the same reason part 3) adds nothing new.

Finally one more lemma for later use in applying the structure equations to surfaces in Euclidean space.

Lemma 3.3.2. (Cartan's lemma)
Suppose $\beta^1 \dots \beta^r \in V^*$ *are linearly independent and* $r \leq n$. *If there are* $\theta^1 \dots \theta^r \in V^*$ *such that* $\sum_{i=1}^r \beta^i \wedge \theta^i = 0$ *then there exist coefficients* $h_j^i \in \mathbb{R}$ *with* $h_j^i = h_i^j$ *such that* $\theta^i = \sum_{j=1}^r h_j^i \beta^j$.

Proof. Extend the β^i to a basis of V^* by introducing new dual vectors $\beta^{r+1}, \dots \beta^n$. Then $\theta^i = \sum_j h_j^i \beta^j$. Using the assumption we have $0 = \sum_{i=1}^r \beta^i \wedge \theta^i = \sum_{i=1}^r \sum_{j=1}^n \beta^i \wedge h_j^i \beta^j$. Since the $\beta^i \wedge \beta^j$ with $i < j$ are linearly independent it follows that $h_j^i = h_i^j$ for $i < j \leq r$ and $h_j^i = 0$ when $j > r$. ☐

For example if $V = \mathbb{R}^3$ and $r = 2$ then we could take $\beta^1 = \varepsilon^1 + \varepsilon^2$ and $\beta^2 = \varepsilon^2 + \varepsilon^3$ and $\theta^1 = 2\varepsilon^1 + 3\varepsilon^2 + f\varepsilon^3$ and $\theta^2 = a\varepsilon^1 + b\varepsilon^2 + c\varepsilon^3$. The condition $\beta^1 \wedge \theta^1 + \beta^2 \wedge \theta^2 = 0$ in the Cartan lemma can then be expanded in terms of the ε^i basis and as in the proof we set $\beta^3 = \varepsilon^3$. Using the short hand $\varepsilon^{ij} = \varepsilon^i \wedge \varepsilon^j$ we find

$$0 = \beta^1 \wedge \theta^1 + \beta^2 \wedge \theta^2 = \varepsilon^{12} + f\varepsilon^{13} + f\varepsilon^{23} - a\varepsilon^{12} - a\varepsilon^{13} + (c - b)\varepsilon^{23} = (1 - a)\varepsilon^{12} + (f - a)\varepsilon^{13} + (f + c - b)\varepsilon^{23}$$

It follows that $1 = f = a = b - c$ because the ε^{ij} with $i < j$ form a basis for $\Lambda^2((\mathbb{R}^3)^*)$. In this case we would indeed have $\theta^1 = 2\beta^1 + f\beta^2$ and $\theta^2 = a\beta^1 + c\beta^2$ with $h_1^2 = a = f = h_2^1$ as claimed.

Exercises

1. Compute the connection 1-forms for the Euclidean frame $X_1, \dots X_n : \mathbb{R}^n \to \mathbb{R}^n$ defined by $X_i(p) = e_i$.
2. Write down explicitly what the structure equations say in case $\dim V = 2$ and run through the arguments in the proofs of this section in this special case.
3. Suppose $\dim V = 2$ and we have an orthonormal frame $X_1, X_2 : P \to V$.
   (a) Show that there exists an element $J \in O(V)$ such that for all $p \in P$ we have $J(X_1(p)) = X_2(p)$ and $J(X_2(p)) = -X_1(p)$.
   (b) Is J in $O^+(V)$ or in $O^-(V)$? What is $J'(p)(v)$? and J^2?

(c) Use the chain rule to prove that $X_2'(p)(v) = J \circ X_1'(p)(v)$.

(d) Apply J to both sides of the defining equation $X_1'(p)(v) = \omega_1^1(p)(v)X_1(p) + \omega_1^2(p)(v)X_2(p)$ to give an alternative proof of $\omega_i^j = -\omega_j^i$.

(e) Prove that $J^*\xi^1 = -\xi^2$ and $J^*\xi^2 = \xi^1$.

(f) Apply J^* to both sides of the structure equation $d\xi^1 = \xi^2 \wedge \omega_2^1$. Does this imply $d\xi^2 = \xi^1 \wedge \omega_1^2$?

4. In this exercise we make a moving frame on an open subset $P \subset \mathbb{R}^n = V$ using a single differentiable unit vector field $m : P \to V$, so $|m(p)| = 1$ for all $p \in P$. Define $X_i(p) = R_{m(p)}e_i$ where $R_m \in O(V)$ denotes the reflection in the mirror orthogonal to m.

   (a) Write an explicit formula for $X_i(p)$ in terms of m and p.

   (b) Take $m(p) = \frac{p}{|p|}$ and $P = \mathbb{R}^n - \{0\}$ and show that $\xi^i(p) = \sum_{j=1}^n \langle X_i(p), e_j\rangle \varepsilon^j$.

   (c) Find explicit expressions for the coframe $\xi^i : P \to V^*$ in terms of ε^j.

   (d) Compute the connection 1-form ω_1^2 in terms of the ξ^j.

   (e) Compute $d\omega_2^1$ explicitly.

5. Given a differentiable map $J : P \to O(V) \subset \mathrm{Hom}(V, V)$ and basis $\mathbf{b}$ of V we define a frame on $P \subset V$ by $X_i(p) = J(p)b_i$. Show that $\omega_i^j(p)(v) = \beta^j(J(p)^{-1}J'(p)(v)b_i)$ where β is the dual basis to $\mathbf{b}$.

6. More on Cartan's lemma

   (a) What does the Cartan lemma claim when $r = 1$?

   (b) Take $V = \mathbb{R}^4$ and $\beta^1 = \varepsilon^1 - 2\varepsilon^2 + \varepsilon^3 + \varepsilon^4$ and $\beta^2 = 2\varepsilon^1 - 4\varepsilon^2 - \varepsilon^3 + \varepsilon^4$ and $\theta^1 = -\varepsilon^1 + 2\varepsilon^2 + 2\varepsilon^3$ and $\theta^2 = -3\varepsilon^1 - 6\varepsilon^2 - 3\varepsilon^3 + \varepsilon^4$. Check that β^1, β^2 are independent and check that $\beta^1 \wedge \theta^1 + \beta^2 \wedge \theta^2 = 0$. Find the coefficients h_j^i that relate the θ in terms of the β.

   (c) If the $\beta^1, \dots \beta^r$ in the Cartan lemma are linearly dependent what will go wrong in the proof? Give an example.

   (d) Suppose there exist coefficients $h_j^i \in \mathbb{R}$ with $h_j^i = h_i^j$ such that $\theta^i = \sum_{j=1}^r h_j^i \beta^j$. Is it true that $\sum_{i=1}^r \beta^i \wedge \theta^i = 0$?

   (e) Suppose that the $\theta^1, \dots \theta^r$ are also linearly independent, then we would find coefficients k_j^i such that $\beta^i = \sum_{j=1}^r k_j^i \theta^j$. What is the relation between the k_j^i and the h_j^i?

3.4 Surfaces in Euclidean space, geodesics and covariant derivative

In this section we will apply the moving frame to study surfaces in Euclidean space W. Throughout this section W will be an m-dimensional inner product space and $P \subset V$ will be open as usual. By a surface we mean the following.

Definition 3.4.1. (Parametrized surface)
A (parametrized) surface is an injective C^2-differentiable map $\varphi : P \to W$ whose derivative is also injective at every point. The tangent space at point $q = \varphi(p)$ is defined as $T_q\varphi = \varphi'(p)(V)$.

A simple way to build examples of parametrized surfaces is to construct them as the graph of a function $f : P \to U$. Setting $W = V \times U$ we define $\varphi(p) = (p, f(p))$, which is C^2 when f is and always is injective with injective derivative at every point (Exercise). The implicit function theorem tells us that this example is quite general in that any level set can be parametrized locally as the graph of a function.

Another example is the parametrization of the unit sphere by spherical coordinates $P = (0, \pi) \times (-\pi, \pi)$ and $\varphi : P \to \mathbb{R}^3$ given by $\varphi(\mu, \lambda) = \sin(\mu)(\cos\lambda e_1 + \sin\lambda e_2) + \cos(\mu)e_3$. At point $p = (\mu, \lambda)$ the tangent space $T_{\varphi(p)}\varphi$ is spanned by the vectors $\partial_\mu\varphi(p) = \cos(\mu)(\cos\lambda e_1 + \sin\lambda e_2) - \sin\mu e_3$ and $\partial_\lambda\varphi(p) = \sin(\mu)(-\sin\lambda e_1 + \cos\lambda e_2)$.

A basic question on surfaces is to find the straightest possible path(s) between two points on the surface. Geodesics are meant as a generalization of the straight lines of Euclidean geometry. What makes a path straight? One way to approach this is to parametrize the path look at its acceleration. For Euclidean straight lines the acceleration must be 0 but our condition on the surface is weaker. We require the acceleration to only occur orthogonal to the tangent plane. Intuitively the acceleration is just there to make sure the curve remains on the surface. Such a phenomenon already occurs in the curve walking on the unit circle $\gamma(t) = (\cos t, \sin t)$ surely this must be a geodesic for the circular surface in $\mathbb{R}^2$ but its acceleration is non-zero, it is in fact $\ddot{\gamma}(t) = -\gamma(t)$.

Recall that for a differentiable curve $\gamma : (-a, a) \to P$ the velocity is $\dot{\gamma} : (-a, a) \to V$ is defined by $\dot{\gamma}(t) = \gamma'(t)(e_1) = \partial_{e_1}\gamma(t)$. When more convenient we will also use the notation $\partial_1\gamma = \dot{\gamma}$. The acceleration is $\ddot{\gamma} : (-a, a) \to V$ defined by $\partial_1\dot{\gamma}$.

Definition 3.4.2. (Geodesic)
A **geodesic** *in a parametrized surface* $\varphi : P \to W$ *is a curve* $\gamma : (-a, a) \to P$ *such that the composite* $\delta = \varphi \circ \gamma$ *satisfies* $\ddot{\delta}(t) \perp T_{\delta(t)}\varphi$ *for all* t.

In the example of the parametrized sphere the meridian $\gamma(t) = (t, 0)$ is an example of a geodesic since if we set $\delta = \varphi \circ \gamma$ then $\dot{\delta}(t) = \partial_1\varphi(t, 0) = \cos(t)e_1 - \sin(t)e_3$ so $\ddot{\delta}(t) = -\sin(t)e_1 - \cos(t)e_3$. The tangent space at point $\varphi(t, 0)$ is spanned by $\partial_1\varphi(t, 0) = \cos(t)e_1 - \sin(t)e_3$ and $\partial_2\varphi(p) = \sin(t)e_2$ so it is clear that our $\ddot{\delta}(t)$ is orthogonal to that. In contrast the longitude $\gamma(t) = (\frac{\pi}{4}, t)$ is NOT a geodesic since $\ddot{\delta}$ is not orthogonal to the tangent plane at $\delta(t)$. Indeed $\delta(\frac{\pi}{4}, t) = \frac{\sqrt{2}}{2}(\cos te_1 + \sin te_2) + \frac{\sqrt{2}}{2}e_3$ so $\ddot{\delta}(t) = \frac{\sqrt{2}}{2}(-\cos te_1 - \sin te_2)$ has inner product $-\frac{1}{2}$ with $\partial_1\varphi(\frac{\pi}{4}, t)$.

Generalizing the situation for geodesics slightly we are interested in computing the following new type of derivative called covariant derivative.

Definition 3.4.3. (Vector field along curve and covariant derivative)
Given a curve $\gamma : (-a, a) \to P$ *we say* $Z : (-a, a) \to V$ *is a vector field along* γ. *Define another vector field* $D(Z)$ *along the same curve by* $\varphi'(\gamma(t))D(Z)(t) = \pi(t)\partial_t\varphi'(\gamma(t))(Z(t))$, *where* $\pi(t)$ *is the orthogonal projection onto the tangent space at* $\varphi(\gamma(t))$.

The velocity $\dot{\gamma}$ is an example of a vector field along the curve γ. In terms of our covariant derivative, the condition for γ to be a geodesic is precisely $D\dot{\gamma} = 0$. Also notice that injectivity of the derivative assures us that the covariant derivative is well-defined. More generally the equation $DZ = 0$ also has geometrical meaning:

Definition 3.4.4. (Parallel vector field)
A vector field Z *along curve* γ *is said to be parallel if* $DZ = 0$.

We think of the vectors of a parallel vector field $Z(t)$ as constant as we walk along the curve. The covariant derivative makes sure that the vectors in a parallel vector field are always perpendicular to the surface. For a geodesic γ the velocity vector field is parallel along γ. More generally if $\dim V = 2$ then we will show that any vector field along a geodesic that has constant length and constant angle with $\dot{\gamma}$ must be parallel. Unlike in the Euclidean case, the notion of parallel usually depends on the curve.

Coming back to the curve $\gamma(t) = (\frac{\pi}{4}, t)$ in our spherical example, we know that $\dot{\gamma}$ is not parallel along γ because it is not a geodesic. A vector field that is in fact parallel along γ is $Z(t) = (\frac{1}{\sqrt{2}}\sin(\frac{t}{\sqrt{2}}), \cos\frac{t}{\sqrt{2}})$.

Covariant derivatives are closely related to the connection 1-forms coming out of a moving frame as we will now see. To keep the exposition simple we will assume from this point on that W is three-dimensional and V has dimension two. The same formulas will however also work in higher dimensions.

We will work with a moving frame X_1, X_2, X_3 defined on a neighborhood $Q \subset W$ containing $\varphi(P)$. Moreover we will assume that our frame is **adapted** to the surface φ in the sense that $X_1(\varphi(p))$ and $X_2(\varphi(p))$ form a basis for the tangent space $T_{\varphi(p)}\varphi$ at point $\varphi(p) \in Q$. Adapted orthonormal frames are easy to build from φ. One way to do it is to apply Gram-Schmidt to the vector fields $\partial_1\varphi(p)$ and $\partial_2\varphi$ to get X_1 and X_2 and finally set $X_3 = X_1 \times X_2$. Of course this is by no means the only option, adapted frames are far from unique.

For example in our spherical coordinates we may choose the following adapted frame $X_i : Q \to W$ on a suitable open subset $Q =\subset W = \mathbb{R}^3$ where say $Q = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 > 0\}$. Using $q = (x, y, z) = xe_1 + ye_2 + ze_3$ we set $X_3(q) = \frac{q}{|q|}$, $X_2(q) = \frac{(-y,x,0)}{\sqrt{x^2+y^2}}$ and $X_1(q) = X_2(q) \times X_3(q) = \frac{(xz,yz,-x^2-y^2)}{|q|\sqrt{x^2+y^2}}$. It should be clear that this frame is adapted to the sphere in that X_3 spans the normal direction. In fact we have $X_2(\varphi(\mu, \lambda)) = \varphi'(\mu, \lambda)(\frac{e_2}{\sin\mu})$ and $X_1(\varphi(\mu, \lambda)) = \varphi'(\mu, \lambda)(e_1)$.

We recall the dual coframe $\xi^1, \xi^2, \xi^3 : Q \to W$ and also consider their pull-backs $\eta^i = \varphi^*\xi^i$ and the connection 1-forms ω_i^j and their pull-backs $\alpha_i^j = \varphi^*\omega_i^j$. The fact that the frame is adapted translates into $\eta^3 = 0$. This is because $\eta^3(p)(v) = \xi^3(\varphi(p))(\varphi'(p)(v)) = 0$, where the last equality holds because $\varphi'(p)(v) \in T_{\varphi(p)}\varphi = \mathrm{Span}\{X_1(\varphi(p)), X_1(\varphi(p))\}$. The other 1-forms η^1, η^2 form a coframe for (P, φ^*g_{Eucl}) and we denote the corresponding frame of P by $Y_1, Y_2 : P \to V$. Notice that $\varphi'(p)Y_i(p) = X_i(\varphi(p))$ by the definition of pull-back and duality (Exercise!).

Suppose $Z : (-a, a) \to V$ is a vector field along $\gamma : (-a, a) \to P$. To compute its covariant derivative first write $Z(t) = Z^1(t)Y_1(\gamma(t)) + Z^2(t)Y_2(\gamma(t))$. To save space we will use the notation $p = \gamma(t)$ and $q = \varphi(p)$. For fixed $i = \{1, 2\}$ we have

$$\varphi'(\gamma(t))(Z^i(t)Y_i(\gamma(t))) = Z^i(t)\varphi'(p)Y_i(p)) = Z^i(t)X_i(q)$$

Differentiating with respect to t gives

$$\partial_t\varphi'(\gamma(t))(Z^i(t)Y_i(\gamma(t))) = \dot{Z}^i(t)X_i(q) + Z^i(t)X_i'(q)(\varphi'(p)\dot{\gamma}(t))$$

Applying the orthogonal projection π and expressing X_i' in terms of connection 1-forms using $\pi X_i'(q)(v) = \omega_i^j(q)(v)X_j(q)$, $\{i, j\} = \{1, 2\}$ we find

$$\pi(t)\partial_t\varphi'(\gamma(t))(Z^i(t)Y_i(\gamma(t))) = \dot{Z}^i(t)X_i(q) + Z^i(t)\omega_i^j(q)\big(\varphi'(p)\dot{\gamma}(t)\big)X_j(q) =$$

$$\dot{Z}^i(t)X_i(q) + Z^i(t)\alpha_i^j(p)\big(\dot{\gamma}(t)\big)X_j(q)$$

We conclude by turning the $X_i(q)$ into $Y_i(p)$ and collecting them:

$$D(Z)(t) = \big(\dot{Z}^1(t) + Z^2(t)\alpha_2^1(p)(\dot{\gamma}(t))\big)Y_1(p) + \Big(\dot{Z}^2(t) + Z^1(t)\alpha_1^2(p)(\dot{\gamma}(t))\Big)Y_2(p) \qquad p = \gamma(t) \tag{3.2}$$

Notice that in the equation for the covariant derivative only two of the connection 1-forms appear: $\alpha_1^2 = -\alpha_2^1$. The other connection forms α_1^3, α_2^3 do not. They will however make an appearance in the investigation of curvature that we turn to next.

Gaussian curvature. Let us investigate how our surface $\varphi : P \to Q \subset W$ is curved by looking how a normal vector field would turn and twist as we walk around the surface. We start with a differentiable unit normal vector field $G : \varphi(P) \to W$ so $|G(\varphi(p))| = 1$ for all $p \in P$. Then $G'(q) : W \to W$ is a linear map that sends $T_{\varphi(p)}\varphi = G(\varphi(p))^\perp$ to itself. Indeed, any $w \in T_{\varphi(p)}$ is of the form $\varphi'(p)(v)$ so

$$\langle G(\varphi(p)), w\rangle = \langle G(\varphi(p)), G'(\varphi(p))(\varphi'(p)(v))\rangle = \langle G(\varphi(p)), (G\circ\varphi)'(p)(v)\rangle = \frac{1}{2}(|G\circ\varphi|^2)'(p)(v) = 0$$

since the length of G is always 1 on the image of φ.

Definition 3.4.5. (Gauss curvature of a parametrized surface)
The Gauss curvature $\kappa : \varphi(P) \to \mathbb{R}$ of parametrized surface $\varphi : P \to W$ with normal vector field G is defined by $\kappa(\varphi(p)) = \det G'(\varphi(p))|_{T_\varphi(p)\varphi}$.

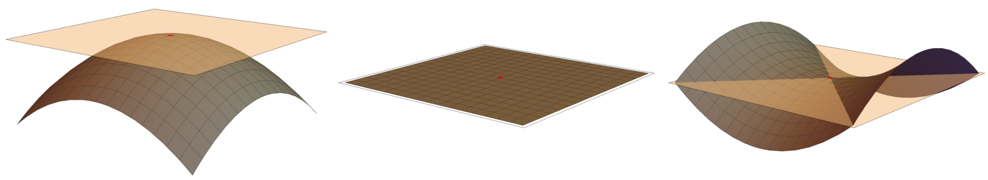

Figure 3.1: The image of $\varphi(x, y) = (x, y, r_1x^2 + r_2y^2)$ with the point 0 in red and the tangent plane at 0 (it is horizontal) for $(r_1, r_2) = (-\frac{1}{2}, -\frac{1}{2})$, $\kappa(0) = 1$ (left), for $(r_1, r_2) = (0, 0)$, $\kappa(0) = 0$ (middle), for $(r_1, r_2) = (\frac{1}{2}, -\frac{1}{2})$, $\kappa(0) = -1$ (right).

For example let us compute $\kappa(0)$ when $\varphi(x, y) = (x, y, r_1x^2 + r_2y^2)$. Setting $p = (x, y)$ the tangent space $T_{\varphi(p)}\varphi$ at $\varphi(p)$ is spanned by $\varphi'(p)(e_1)$ and $\varphi'(p)(e_2)$. Taking the cross product we have a normal vector

$$G(\varphi(p)) = N(\varphi'(p)(e_1) \times \varphi'(p)(e_2)) = \frac{(-2r_1x, -2r_2y, 1)}{\sqrt{4r_1^2x^2 + 4r_2^2y^2 + 1}}$$

To compute $\kappa(0)$ we set $b_i = \varphi'(0)(e_i)$ and use the chain rule to get $G'(\varphi(0))(b_i) = (G\circ\varphi)'(0)(e_i) = -2r_ib_i$. Therefore the matrix of $G'(\varphi(0))|_{T_0\varphi}$ with respect to the basis b_1, b_2 is diagonal and its determinant is $4r_1r_2$. We conclude the curvature at 0 is $\kappa(0) = 4r_1r_2$.

We notice that choosing the other normal vector $-G$ would give the same Gauss curvature and κ only depends on the surface $\varphi(P)$, not the parametrization itself. Also this example illustrates a general phenomenon that when $\kappa(q) > 0$ the tangent plane close to q is on one side of the surface while when $\kappa(q) < 0$ the tangent plane passes through as a saddle does in the picture. The case $\kappa(q) = 0$ represents a surface that coincides with the tangent plane at q in at least a line.

Computing the Gauss curvature directly in more complicated cases will get quite messy. Below we illustrate the power of the method of the moving frame and Cartan's structure equations to give an alternative computation of the curvature. Our starting point will be an adapted frame $X_1, X_2, X_3 : Q \to W$ as above. X_3 now plays the role of the unit normal $G = X_3$ while X_1, X_2 span the tangent plane at each point. We also define $M(p) = X_3 \circ \varphi(p)$. The chain rule tells us that if we set $q = \varphi(p)$ and $w = \varphi'(p)(v)$ we have

$$M'(p)(v) = X_3'(q)(w) = \omega_3^1(q)(w)X_1(q) + \omega_3^2(q)(w)X_2(q) = \alpha_3^1(p)(v)X_1(q) + \alpha_3^2(p)(v)X_2(q)$$

Here as above we set $\alpha_i^j = \varphi^*\omega_i^j$. To make further progress we use the structure equations to express the α's in terms of the $\eta^i = \varphi^*\xi^i$.

Since the frame is adapted we have $0 = \eta^3$ so we must also have $d\eta^3 = 0$. Using the fact that d and pull-back can be interchanged we can now apply the second structure equation 3.3.1 to get

$$0 = d\eta^3 = d\varphi^*\xi^3 = \varphi^*d\xi^3 = \varphi^*\sum_k \xi^k \wedge \omega_k^3 = \sum_k \varphi^*\xi^k \wedge \varphi^*\omega_k^3 = -\sum_k \eta^k \wedge \alpha_3^k$$

Using Cartan's Lemma 3.3.2 we thus find $\alpha_3^1 = h_1^1\eta^1 + h_2^1\eta^2$ and $\alpha_3^2 = h_1^2\eta^1 + h_2^2\eta^2$ for some functions h_j^i satisfying $h_2^1 = h_1^2$.

Now set up a basis for the tangent space $b_i = X_i(\varphi(p)) = \varphi'(p)(Y_i(p))$ for $i \in \{1,2\}$, with Y_i dual to η^i. We have $G'(\varphi(p))(b_i) = \alpha_3^1(p)(Y_i(p))b_1 + \alpha_3^2(p)(Y_i(p))b_2$ so the matrix of $G'(\varphi(p))|_{T_q\varphi}$ with respect to the b_1, b_2 basis is $\begin{pmatrix} h_1^1 & h_2^1 \\ h_1^2 & h_2^2 \end{pmatrix}$. The determinant is thus $\kappa(\varphi(p)) = h_1^1(p)h_2^2(p) - h_1^2(p)h_2^1(p)$.

An even easier way to obtain the same Gauss curvature is through the third structure equation. We simply compute

$$d\alpha_2^1 = d\varphi^*\omega_2^1 = \varphi^* d\omega_2^1 = \varphi^*(\omega_2^3 \wedge \omega_3^1) = \alpha_2^3 \wedge \alpha_3^1 = -(h_1^2\eta^1 + h_2^2\eta^2) \wedge (h_1^1\eta^1 + h_2^1\eta^2) = \kappa\eta^1 \wedge \eta^2$$

A remarkable fact about the Gauss curvature is that it only depends on the surface itself, not on the choice of parametrization of the surface and not even on the sign of the unit normal vector. Even more surprisingly the Gauss curvature does not even depend on the way the surface sits in space, just on the pull-back metric $\varphi^* g_{Eucl}$. It is an intrinsic quantity. We will have more to say about this in the next section but briefly it is because α_1^2 is determined by η^1 and η^2 and the pull-back metric alone. Indeed we have $d\eta^1 = \eta^2 \wedge \alpha_2^1$ and $d\eta^2 = \eta^1 \wedge \alpha_1^2$. It follows that $\alpha_1^2 = f_1\eta^1 + f_2\eta^2$ with $f_i(p) = (d\eta^i)(p)(Y_1(p) \wedge Y_2(p))$.

For example in the parametrized sphere we set up an adapted frame X_1, X_2, X_3 and we have a corresponding orthonormal frame $Y_1, Y_2 : P \to V$ defined by $Y_1(\mu, \lambda) = e_1$ and $Y_2(\mu, \lambda) = \frac{e_2}{\sin\mu}$. The corresponding co frame is $\eta^1(\mu, \lambda) = \varepsilon^1$ and $\eta^2(\mu, \lambda) = \sin(\mu)\varepsilon^2$. Their exterior derivatives are $d\eta^1(\mu, \lambda) = 0$ and $d\eta^2(\mu, \lambda) = \cos(\mu)\varepsilon^1 \wedge \varepsilon^2 = \cot(\mu)\eta^{12}$. Therefore the connection 1-form must be $\alpha_1^2(\mu, \lambda) = \cot(\mu)\eta^2$. The Gauss curvature is then computed as $d\alpha_1^2(\mu, \lambda) = -\sin(\mu)\varepsilon^{12} = -\eta^1 \wedge \eta^2$. In conclusion the curvature is 1 at every point.

Exercises

1. Consider $P = \mathbb{R}^2$ and $\varphi : P \to \mathbb{R}^3 = W$ defined by $\varphi(a, b) = (a, b, a^2 - b^2)$.
   (a) Find an adapted orthonormal frame X_1, X_2, X_3 on W, so that X_3 is normal to the surface $\varphi(P)$ and $\varphi'(p) = \text{Span}(X_1(p), X_2(p))$.
   (b) Compute the dual 1-forms ξ^1 and ξ^2
   (c) Compute their pull-backs $\eta^1 = \varphi^*\xi^1$ and $\eta^2 = \varphi^*\xi^2$.
   (d) Find out how to express α_1^2 in terms of η^1, η^2.
   (e) Compute the Gaussian curvature by expressing $d\alpha_1^2$ in terms of $\eta^1 \wedge \eta^2$.
2. Imagine a part of a cylinder in $W = \mathbb{R}^4$ parametrized by $\varphi : P \to \mathbb{R}^4$ where $P = (-\pi, \pi) \times (-1, 1)$ and $\varphi(a, b) = b(e_1 + e_3) + \cos(a)e_2 + \sin(a)e_4$.
   (a) Find a basis for the tangent plane $T_{\varphi(p)}\varphi$ at the point $p = (a, b)$ and write it down explicitly in terms of e_1, e_2, e_3, e_4.
   (b) Verify that the curve $\gamma(t) = (t, t)$ defines a geodesic curve and $\beta(t) = (t^2, 0)$ does not.
3. Surface of revolution. Imagine a positive C^2 differentiable function $h : (-1, 1) \to (0, \infty)$ and consider a parametrized surface $\varphi : P \to \mathbb{R}^3$ that represents the result of rotating the graph of h around the x-axis. We have $P = (-1, 1) \times (0, 2\pi)$ and $\varphi(s, t) = se_1 + h(s)(e_2\cos(t) + e_3\sin(t))$.
   (a) Describe the tangent plane at any point $\varphi(p)$ explicitly.
   (b) Find an expression for a normal vector at every point of the surface.
   (c) Is it true that for any constant c the curve $\gamma : (-1, 1) \to P$ defined by $\gamma(t) = (t, c)$ is a geodesic? What about the curve $\gamma(t) = (c, t)$?
4. Consider a parametrized surface $\varphi : P \to W$ in inner product space W with an adapted frame $X_1, \ldots X_n : Q \to W$ where Q is an open subset containing $\varphi(P)$ (meaning $X_1, \ldots X_r$ span the tangent space) and $\xi^1 \ldots \xi^n$ is the corresponding coframe. Prove that any 1-form α on P can be written as $\alpha = \sum_{i=1}^r f_i\eta^i$ for some functions $f_i : P \to \mathbb{R}$ and $\eta^i = \varphi^*\xi^i$. Also show that if $Y_1, \ldots, Y_r$ form an orthonormal frame (with respect to the pull-back metric) then $\varphi'(p)Y_i(p) = X_i(\varphi(p))$.
5. Prove that the speed of a geodesic $\gamma : (-a, a) \to P$ in a parametrized surface $\varphi : P \to W$ is constant. More precisely prove that $|\dot{\delta}(t)|$ does not depend on time, where as usual $\delta = \varphi \circ \gamma$.
6. We view the sphere as parametrized by the stereographic projection $\varphi : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $\varphi(x, y) = \frac{(2xe_1 + 2ye_2 + (-1 + x^2 + y^2)e_3)}{1 + x^2 + y^2}$. You may freely use (or check for yourself) the fact that
   $\varphi'(x, y)(e_1) = \frac{2(1 - x^2 + y^2)e_1 - 4xye_2 + 4xe_3}{(1 + x^2 + y^2)^2}$ and $\varphi'(x, y)(e_2) = \frac{-4xye_1 + 2(1 + x^2 - y^2)e_2 + 4ye_3}{(1 + x^2 + y^2)^2}$.

(a) If $g = \varphi^* g_{Eucl}$, compute $g_{ij}(x, y) = g(x, y)(e_i, e_j)$.

(b) Find an orthonormal frame and an orthonormal coframe for the Riemannian chart $(\mathbb{R}^2, g)$.

(c) Compute the Gaussian curvature at every point.

7. Consider the parametrization $\varphi : P \to \mathbb{R}^3$ where $P = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ and $\varphi(x, y) = (x, y, \sqrt{1 - x^2 - y^2})$. Also imagine the curve $\gamma : (-\pi, \pi) \to P$ defined by $\gamma(t) = \frac{1}{2}(\cos(t), \sin(t))$ and vector field $Z(t) = (\cos(t), \sin(t))$ along the curve γ.

(a) Compute the partial derivatives of φ with respect to x and y and argue that they form a basis for the tangent space $T_q\varphi$ at point $q = \varphi(p)$ for any $p = (x, y) \in P$.

(b) Compute the cross product of your two basis vectors to find a vector normal to the tangent plane at point q.

(c) If $\delta = \varphi \circ \gamma$ the show that $W(t) = \varphi'(\gamma(t))(Z(t))$ is a vector field along curve δ.

(d) Describe the derivative $\dot{W}(t) = \partial_t W(t)$ and also its orthogonal projection G onto the tangent plane at $q = \delta(t)$ so $G(t) = \pi(\delta(t))\dot{W}(t)$.

(e) Find a vector field H along γ such that $\varphi'(\gamma(t))(H(t)) = G(t)$.

(f) Is Z parallel along γ?

3.5 Two-dimensional Riemannian geometry

Motivated by our extrinsic investigation of surfaces in Euclidean space we aim to introduce similar notions purely in the intrinsic setting. First we will show that there exists a 1-form α_1^2 playing the role of connection 1-form. Using the same formula as in the extrinsic case we can still talk about covariant derivative and hence introduce curvature, geodesics and parallelism. Finally we will state and prove a local version of the famous Gauss-Bonnet theorem. Roughly it says that integrating the curvature on a polygon measures the angle sum.

Our starting point will always be two-dimensional Riemannian chart (P, g) where $P \subset \mathbb{R}^2 = V$ is non-empty and open. For simplicity we choose the orientation defined by the standard basis e_1, e_2 at every point. Next we will always choose a positively oriented orthonormal frame $Y_1, Y_2 : P \to V$ with dual coframe $\eta^1, \eta^2 : P \to V^*$. Here orthonormal means orthonormal with respect to g, not the standard Euclidean inner product on $\mathbb{R}^2$. So e_1, e_2 is generally NOT orthonormal.

Besides the Riemannian charts coming from surfaces in $\mathbb{R}^3$ we keep in mind the hyperbolic plane, where we have $P = \mathbb{R} \times (0, \infty)$. If we write $p = (x, y)$ then the metric is $g_{ij}(p) = \frac{1}{y^2}\delta_{ij}$ and can take the orthonormal frame $Y_i(p) = ye_i$. The coframe is $\eta^i(p) = \frac{\varepsilon^i}{y}$.

Our first step is to assert that there is always a unique form α_2^1 that represents the connection 1-form. When our Riemannian chart has a metric that is the pull-back of a Euclidean metric along a parametrized surface φ then $\alpha_2^1 = \varphi^*\omega_2^1$. Here ω is the connection 1-form for an adapted frame X corresponding to our frame Y on the chart.

Lemma 3.5.1. (Levi-Civita connection 1-form)
For any orthonormal frame Y_1, Y_2 there exists a unique 1-form α_1^2 on P such that for $i \in \{1, 2\}$ we have $d\eta^i = \sum_{k=1}^2 \eta^k \wedge \alpha_k^i$. As usual we defined $\alpha_i^j = -\alpha_j^i$.

Proof. We first show uniqueness. Suppose we have such a α_1^2 then we can write $\alpha_1^2 = s_1\eta^1 + s_2\eta^2$ for some functions s_i. The conditions on α_1^2 then imply $d\eta^i = s_i\eta^1 \wedge \eta^2$ for $i = 1, 2$ so if it exists α_1^2 must be unique.

For existence we simply define $s_i = d\eta^i(Y_1 \wedge Y_2)$ for $i = 1, 2$. This satisfies the required equations for example the first equation $d\eta^1 = \eta^2 \wedge \alpha_2^1$. To check this it suffices to see that both sides give the same answer when we evaluate on $Y_1 \wedge Y_2$ at a given point p. And it does because $d\eta^1(Y_1 \wedge Y_2) = s_1 = -s_1(\eta^2 \wedge \eta^1)(Y_1 \wedge Y_2) = (\eta^2 \wedge \alpha_2^1)(Y_1 \wedge Y_2)$. □

For example in the hyperbolic plane we have $d\eta^i(p) = -\frac{1}{y^2}\varepsilon^2 \wedge \varepsilon^i$ so $d\eta^1 = \frac{1}{y^2}\varepsilon^{12} = \eta^1 \wedge \eta^2$ and $d\eta^2 = 0$. It follows that we must take $\alpha_1^2 = \eta^1$.

Definition 3.5.1. (Gauss curvature)
The Gauss curvature $K : P \to \mathbb{R}$ is defined by $d\alpha_2^1(p) = \kappa(p)\eta^{12}$.

In the hyperbolic plane example we have $\alpha_2^1 = -\eta^1$ so $d\alpha_2^1 = -d\eta^1 = -\eta^{12}$ so the curvature is $\kappa = -1$ at every point. This is one feature that makes the hyperbolic plane very special.

By a Z is a vector field along curve $\gamma : (-a, a) \to P$ we just mean a function $Z : (-a, a) \to V$. As in the parametrized surface case we seek to differentiate Z covariantly. Recall that in the parametrized surface case we found an expression for the covariant derivative only involving the connection 1-form. This motivates us to *define*:

Definition 3.5.2. (Covariant derivative, parallel, geodesic)
The covariant derivative of a vector field $Z : (-a, a) \to V$ *along curve* $\gamma : (-a, a) \to P$ *expressed as* $Z(t) = \sum_{i=1}^{2} Z^i(t)Y_i(\gamma(t))$ *is the vector field* $DZ : (-a, a) \to V$ *along* γ *defined by*

$$D(Z)(t) = \left(\dot{Z}^1(t) + Z^2(t)\alpha_2^1(p)(\dot{\gamma}(t))\right)Y_1(p) + \left(\dot{Z}^2(t) + Z^1(t)\alpha_1^2(p)(\dot{\gamma}(t))\right)Y_2(p) \qquad p = \gamma(t)$$

Z *is called parallel along* γ *if* $DZ = 0$ *and* γ *is called a geodesic if* $\dot{\gamma}$ *is parallel along* γ*.*

For example in the hyperbolic plane a curve $\gamma = (\gamma_1, \gamma_2)$ is a geodesic if $Z = \dot{\gamma} = \dot{\gamma}_1 e_1 + \dot{\gamma}_2 e_2 = \frac{\dot{\gamma}_1}{\gamma_2}Y_1(\gamma) + \frac{\dot{\gamma}_2}{\gamma_2}Y_2(\gamma)$ has covariant derviative 0. Since $Z_1 = \frac{\dot{\gamma}_1}{\gamma_2}$ and $Z_2 = \frac{\dot{\gamma}_2}{\gamma_2}$ and $\alpha_1^2(p) = \eta^1(p)$ this simplifies to $\gamma_2\ddot{\gamma}_1 = 2\dot{\gamma}_2\dot{\gamma}_1$ and $\gamma_2\ddot{\gamma}_2 = \dot{\gamma_2}^2 - \dot{\gamma}_1^2$. It can be checked that the solutions are semi-circles with center on the horizonal line $y = 0$ and vertical lines.

From the mere fact that the equations for geodesics and parallel vector fields are systems of ordinary differential equations means we can always solve them (theoretically and numerically).

Theorem 3.5.1. (Existence and uniqueness of geodesics)
For any $p \in P$ *and any* $v \in V$ *there exists a geodesic* $\gamma : (-a, a) \to P$ *with* $\gamma(0) = p$ *and* $\dot{\gamma}(0) = v$ *for some* $a > 0$*. The geodesic is unique in the sense that any other geodesic with the same domain will coincide with it.*

If $v \in V$ *and* $\gamma : (-a, a) \to P$ *is a differentiable curve, then there exists a* $0 < b < a$ *and a parallel vector field* $Z : (-b, b) \to V$ *along* $\gamma|_{(-b,b)}$ *such that* $Z(0) = v$*.*

Proof. Picard's existence and uniqueness theorem for solving ODE tells us that for any vector field $F : Q \to W$ and any $q \in Q$ there exists a unique integral curve $\zeta : (-c, c) \to Q$ through q. In other words we can find a solution to the system of first order differential equations $\dot{\zeta}(t) = F(\zeta(t))$ with initial condition $\zeta(0) = q$. This applies directly to the equations $DZ(t) = 0$ after brining $\dot{Z}$ to one side. In the geodesic equation case we apply the theorem to the curve $\delta : (-a, a) \to P \times V$ defined by $\delta(t) = (\gamma(t), \dot{\gamma}(t))$. This way the second derivatives $\ddot{\gamma}$ are just a part of $\dot{\delta}$. Also the initial condition now becomes $\delta(0) = (p, v)$. □

Lemma 3.5.2. *The covariant derivative is compatible with the Riemannian metric in the following sense:*

$$\frac{d}{dt}g(\gamma(t))(Z(t), W(t)) = g(\gamma(t))(DZ(t), W(t)) + g(\gamma(t))(Z(t), DW(t))$$

for any differentiable vector fields Z, W *along curve* γ*.*

Proof. Writing $Z(t) = \sum_{i=1}^{2} Z^i(t)Y_i(\gamma(t))$ and the same for W and using the orthonormality of the frame the right hand side produces a sum of eight terms, four of which cancel out. The other four make up the left hand side. □

It follows from this lemma that any geodesic must have constant speed $|\dot{\gamma}|_g$, exercise! It also follows that a parallel vector field Z along a geodesic γ must have constant length and constant angle with $\dot{\gamma}$. Indeed, we have $\frac{d}{dt}g(\gamma(t))(Z(t), Z(t)) = 2g(\gamma(t))(Z(t), DZ(t)) = 0$ and $\frac{d}{dt}g(\gamma(t))(Z(t), \dot{\gamma}(t)) = g(\gamma(t))(Z(t), D\dot{\gamma}(t)) + g(\gamma(t))(DZ(t), \dot{\gamma}(t)) = 0$.

The connection 1-form can also be used to measure how far the tangent vector of a geodesic rotates relative to the frame.

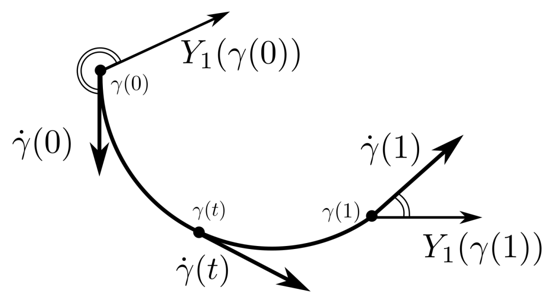

Figure 3.2: Integration of the connection 1-form along a geodesic yields the angles at the endpoints.

Lemma 3.5.3. (Integrating the connection 1-form along a geodesic)
Suppose $\gamma : [0, 1] \to P$ *is a curve that can be reparametezed to be a geodesic and* Y_1, Y_2 *a positive orthonormal frame. We have*

$$\int_\gamma \alpha_2^1 = \angle_{g(\gamma(1))}(Y_1(\gamma(1)), \dot{\gamma}(1)) - \angle_{g(\gamma(0))}(Y_1(\gamma(0)), \dot{\gamma}(0)) \mod 2\pi$$

where $\angle_{g(p)}$ *means the angle with respect to the inner product* $g(p)$*, which is defined modulo* 2π*.*

Proof. The integral of a 1-form over a curve is independent of the parametrization of the curve so we may as well assume that γ is a unit speed geodesic. Since $|\dot{\gamma}|_g = 1$ there must be a differentiable function $\theta : [0,1] \to \mathbb{R}$ such that $\dot{\gamma}(t) = Z^1(t)Y_1(\gamma(t)) + Z^2(t)Y_2(\gamma(t))$, where $Z^1(t) = \cos\theta(t)$ and $Z^2(t) = \sin\theta(t)$. Taking the coefficient of Y_1 in the geodesic condition $D\dot{\gamma} = 0$ tells us $\dot{Z}^1(t) + Z^2(t)\alpha_2^1(\gamma(t))(\dot{\gamma}) = -\sin(\theta(t))\dot{\theta}(t) - \sin(\theta(t))\alpha_1^2(\gamma(t))(\dot{\gamma}) = 0$ so provided $\sin(\theta(t)) \neq 0$ we conclude $\alpha_2^1(\gamma(t))(\dot{\gamma}) = \dot{\theta}(t)$. In case $\sin\theta(t) = 0$ we just use the Y_2 coefficient to obtain the same result (Exercise!). By definition of the integral $\int_\gamma \alpha_2^1 = \int_{[0,1]} \dot{\theta}(t) = \theta(1) - \theta(0)$ using the fundamental theorem of calculus. The result follows. □

Putting everything we learnt together we arrive at the Gauss-Bonnet theorem relating angles to the integral of the curvature. The main idea here is to use the Stokes theorem keeping in mind that $d\alpha_2^1$ equals the curvature times the volume form.

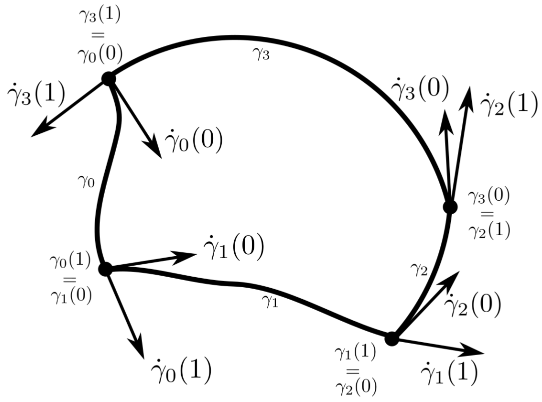

Figure 3.3: The geodesic square parametrized by γ and the tangent vectors at its corners.

Theorem 3.5.2. (Gauss-Bonnet for geodesic squares)
Imagine a differentiable map $\gamma : [0,1]^2 \to P$ such that the curves $\gamma_i : [0,1] \to P$ defined by $\gamma_0(t) = \gamma(t,0)$, $\gamma_1(t) = \gamma(1,t)$, $\gamma_2(t) = \gamma(1-t,1)$ and $\gamma_3(t) = \gamma(0,1-t)$ are geodesics up to reparametrization. If ν is the volume 2-form consistent with the orientation and κ is the Gauss curvature of (P,g) then

$$\int_\gamma \kappa\nu = \sum_{i=0}^{3} \angle_{g(\gamma_i(1))}(\dot{\gamma}_i(1), \dot{\gamma}_{i+1}(0)) \mod 2\pi$$

Proof. Recall the faces of γ are the curves $\gamma_{i,\sigma} : [0,1] \to P$ obtained from γ by setting the i-th coordinate to σ for $\sigma \in \{0,1\}$ and $i \in \{1,2\}$. The boundary of γ is by definition the 1-chain $\partial\gamma = \gamma_{2,0} + \gamma_{1,1} - \gamma_{2,1} - \gamma_{1,0}$.

Choose a positive orthonormal frame Y_1, Y_2 with coframe η^1, η^2 and connection form α_1^2 as usual. The volume 2-form ν equals $\nu = \eta^1 \wedge \eta^2$.

By Stokes theorem and Lemma 3.5.3 we find

$$\int_\gamma \kappa\nu = \int_\gamma \kappa\eta^{12} = \int_\gamma d\alpha_2^1 = \int_{\partial\gamma} \alpha_2^1 = \sum_{i=0}^{3} \int_{\gamma_i} \alpha_2^1 = \sum_{i=0}^{3} \angle_g(\gamma_i(1), \gamma_{i+1}(0))$$

here we noticed that γ_0 is precisely $\gamma_{2,0}$ and $\gamma_1 = \gamma_{1,1}$ while γ_2 and γ_3 coincide with $\gamma_{2,1}(1-t)$ and $\gamma_{1,0}(1-t)$. □

Some striking special cases occur when the curvature κ is constant on the image of our square. In that case we find that the angle sum is κ times the area of the parametrized square. When $\kappa = 0$ it follows that the angle sum is 0. Notice the angles are exterior here so this does agree with the usual statement that the sum of the internal angles in a square is $0 \mod 2\pi$.

The Gauss-Bonnet theorem can be generalized to arbitrary polygons, and then also to surfaces. In case of closed surfaces the integral of the curvature be a constant times the Euler characteristic. Beyond that the Gauss-Bonnet theorem is generalized to arbitrary Riemannian charts and spaces and is a special case of the famous Atiyah-Singer index theorem, one of the greatest achievements of 20th century mathematics.

Finally the reader may object that we used a single orthonormal frame and the connection 1-form and covariant derivative that comes with it. What would happen if we chose a different orthonormal frame instead?

Lemma 3.5.4. (Change of frame)
Imagine another positively oriented orthonormal frame $\tilde{Y}_1, \tilde{Y}_2$ with coframe $\tilde{\eta}^1, \tilde{\eta}^2$ and connection 1-forms $\tilde{\alpha}_i^j$ and corresponding covariant derivative $\tilde{D}$.

1. *There exists a* 1*-form* $\tau : P \to V^*$ *with* $d\tau = 0$ *such that* $\tilde{\alpha}_1^2 = \tau + \alpha_1^2$.

2. *The Gauss curvature is independent of the chosen positive frame.*

3. $DZ = \tilde{D}Z$.

4. *The notions of geodesics and parallelism are independent of the chosen frame.*

Proof. There must be a function $\theta : P \to \mathbb{R}$ such that $\tilde{Y}_1 = \cos(\theta(t))Y_1 + \sin(\theta(t))Y_2$ and this way we can finish the proof by just converting everything into one frame. □

Exercises

1. Suppose $F : P \to P$ is an isometry from the Riemannian chart (P, g) to itself, so $F^*g = g$. Also consider a differentiable curve $\gamma : (-a, a) \to P$.

   (a) Prove that if $Y_1, Y_2 : P \to V$ is an orthonormal frame with respect to g then so is $X_1, X_2 : P \to V$ defined by $X_i(F(p)) = F'(p)(Y_i(p))$.

   (b) Also show that if $\xi^1, \xi^2 : P \to V^*$ and η^1, η^2 are the coframes dual to X_i and Y_i then $F^*\xi^i = \eta^i$.

   (c) Next prove that if α_i^j is the connection 1-form corresponding to Y_1, Y_2 and ω_i^j is the connection 1-form corresponding to frame X_i then $\alpha_i^j = F^*\omega_i^j$.

   (d) Prove that if $\gamma : (-a, a) \to P$ is a geodesic then so is $F \circ \gamma$.

2. Hyperbolic geodesics.

   (a) Show that $F_u(x, y) = u\frac{(-x,y)}{x^2+y^2}$ and $T_r(x, y) = (r + x, y)$ define isometries from hyperbolic space to itself for any $u > 0$ and $r \in \mathbb{R}$.

   (b) Verify that the curve $\gamma(t) = (0, e^t)$ is a hyperbolic geodesic.

   (c) Use the previous exercise to conclude that $T_r \circ F_u \circ T_s \circ \gamma$ is also a hyperbolic geodesic and that these are all parametrizing (Euclidean) semi-circles with (Euclidean) center on the x-axis. Moreover, for any such semi-circle is the image of a hyperbolic geodesic.

   (d) Did we find for every point p and every vector v a unique geodesic passing through p with velocity v?

   (e) Integrate $\frac{\varepsilon^1}{y}$ over part of a semi-circle and explain your finding in terms of angles.

3. Recall a Riemannian metric g on P is conformal if there exists a C^2 differentiable positive function $h : P \to (0, \infty)$ such that $g(p) = h^2(p)g_{Eucl}(p)$, where g_{Eucl} refers to the standard inner product in V.

   (a) Show that there is an orthonormal frame $Y_i(p) = \frac{e_i}{h(p)}$ with dual coframe $\eta^i(p) = h(p)\varepsilon^i$.

   (b) The Euclidean metric is an example of a conformal metric with $h = 1$, do you know any other metrics that are conformal? What is h?

   (c) Apply the proof of Lemma 3.5.1 to find the corresponding connection 1-form α_2^1.

   (d) Also compute $d\alpha_2^1$ and write the Gauss curvature $\kappa(p)$ in terms of h.

   (e) Write down a partial differential equation for h which says that the curvature is constant.

4. Riemannian Review.

   (a) Give an example of a Riemannian chart together with an orthonormal frame whose connection 1-form is 0.

   (b) Explain how to finish the proof of Lemma 3.5.3 in case $\sin\theta(t) = 0$.

   (c) Suppose we have a Riemannian chart (P, g) with $P = \mathbb{R}^2 - \{(0, 0)\}$ and an orthonormal frame Y_1, Y_2 such that the corresponding connection 1-form is $\alpha_2^1(x, y) = \frac{y\varepsilon^1 - x\varepsilon^2}{x^2+y^2}$. What is the curvature?

   (d) If $\gamma : [0, 1] \to P$ is a geodesic with $\gamma(0) = \gamma(1)$ and $\dot{\gamma}(0) = \dot{\gamma}(1)$ is it true that $\int_\gamma \alpha_2^1$ is a multiple of 2π?

3.6 Hodge star, divergence theorem and harmonic functions

Definition 3.6.1. (Hodge star)
The Hodge star on an oriented (P, g) *is the map* $\Omega^k(P) \xrightarrow{\star} \Omega^{n-k}(P)$ *defined pointwise by* $(\star\omega)(p) = \star(\omega(p))$.

For example the pull-back of the Euclidean metric in spherical coordinates (μ, λ) had orthonormal basis $b_1(\mu, \lambda) = e_1, b_2(\mu, \lambda) = (\sin\mu)^{-1}e_2$ at point (μ, λ). With dual basis $\beta^1(\mu, \lambda) = \varepsilon^1$ and $\beta^2 = \sin\mu\varepsilon^2$. Therefore the Hodge star is $\star\beta^1 = \beta^2$.

The Hodge star allows us to formulate some of the fundamental partial differential equations in a truely geometric way. Many of the famous fundamental partial differential equations involve the Laplacian Δ. For example the Laplace equation $\Delta u = 0$, the Poisson equation $\Delta u = f$, the heat equation, the wave equation, the diffusion equation, the Schödinger equation, the Klein-Gordon equation, the Helmholtz equation, the Maxwell equation and the Navier-Stokes equation.

Definition 3.6.2. (Hodge Laplacian)
A choice of metric g *and orientation on* P *defines for all* k *the Laplacian* $\Omega^k(P) \xrightarrow{\Delta^k} \Omega^k(P)$ *by* $\Delta^k\omega = (\star d \star d + d \star d\star)\omega$

As a basic example take $\mathbb{R}^n$ with the Euclidean metric. Viewing a C^2 function $\mathbb{R}^n \xrightarrow{f} \mathbb{R}$ as an element of $\Omega^0(P)$ we see that $\star f = f[e] \in \Omega^n(\mathbb{R}^n)$ and so $d \star f = 0$. Also $df = \sum_i \frac{\partial f}{\partial x_i} e^1$ so $\star df = \sum_i \frac{\partial f}{\partial x_i}(-1)^i \bigwedge_{j\neq i} e^i$ and $d \star df = \sum_i \frac{\partial^2 f}{(\partial x_i)^2}[e]$ and so $\Delta^0 f = \sum_i \sum_i \frac{\partial^2 f}{(\partial x_i)^2}$. So our Laplacian is a very complicated (but geometrically natural!) generalization of the sum of the second partial derivatives.

If we allow ourselves to use the Minkowski metric on $P = \mathbb{R}^4$ (not positive definite) then the equation $\Delta^2 = 0$ is precisely Maxwell's equations (in the absence of charges). Also $\Delta^0 = 0$ is the wave equation.

The famous Hodge conjecture (one of the million dollar problems) basically says that all harmonic k-covector fields with rational coefficients on a complex non-singular complex manifold correspond to complex submanifolds of dimension k. So what are manifolds?