Linear Algebra cheat sheet

1 Vector spaces

In the following $k = \mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$.

Definition. A k**-vector space** is an abelian group V together with a map $k \times V \stackrel{\cdot}{\longrightarrow} V$, $(\lambda, v) \mapsto \lambda v$, satisfying

1. $\lambda(v_1 + v_2) = \lambda v_1 + \lambda v_2$,
2. $(\lambda_1 + \lambda_2)v = \lambda_1 v + \lambda_2 v$,
3. $(\lambda\mu)v = \lambda(\mu v)$,
4. $1 \cdot v = v$,

for all $\lambda, \mu, \lambda_1, \lambda_2 \in k$ and all $v, v_1, v_2 \in V$.
A subgroup $U \subseteq V$ is a **linear subspace** if $\lambda u \in U$ for all $\lambda \in k, u \in U$.

Definition. A group homomorphism $f : V \longrightarrow W$ between vector spaces is k**-linear** if

$$f(\lambda v) = \lambda f(v)$$

for all $\lambda \in k, v \in V$.
If f is bijective, it is a **linear isomorphism**, and in such a case so if f^{-1}.
The composite of linear maps is also linear.

Definition. Let $f : V \longrightarrow W$ be a linear map. The **kernel** and the **image** of f are

$$\operatorname{Ker} f := \{v \in V : f(v) = 0\},$$
$$\operatorname{Im} f := \{f(v) \in W : v \in V\},$$

and both are linear subspaces (of V and W, respectively).

Lemma 1.1 *A linear map $f : V \longrightarrow W$ is injective if and only if* $\operatorname{Ker} f = 0$.

Recall that if G is a group and $H \subseteq G$ is a normal subgroup, the factor group G/H (also known as the **quotient group**) is the set of subsets of the form $g + H = \{g + h : h \in H\}$, and this set is a group with group law $(a + H) + (b + H) := (a + b) + H$. Observe that here additive (and not multiplicative) notation for the group law has been used.

Proposition 1.2 *Let V be a vector space and $U \subseteq V$ a linear subspace. The quotient group V/U has a structure of k-vector space defined by $\lambda(v + U) := (\lambda v) + U$, where $\lambda \in k, v \in V$.*

Theorem 1.3 *Let $f : V \longrightarrow W$ be a linear map. Then there is an isomorphism of vector spaces*

$$\bar{f} : V/\operatorname{Ker} f \xrightarrow{\cong} \operatorname{Im} f\ ,\ \bar{f}(v + \operatorname{Ker} f) := f(v).$$

2 Dimension theory

Any ordered sequence of vectors $(b_1, \ldots, b_n)$ in V defines a linear map

$$\mathbf{b} : k^n \longrightarrow V\ ,\ \mathbf{b}(x_1, \ldots, x_n) = x_1 b_1 + \cdots + x_n b_n.$$

Definition.

1. We say that the vectors $(b_1, \ldots, b_n)$ are **linearly independent** when $\mathbf{b}$ is injective, that is, when the only null linear combination $x_1 b_1 + \cdots + x_n b_n = 0$ is the trivial one, $x_1 = \cdots = x_n = 0$.
2. We say that the vectors $(b_1, \ldots, b_n)$ **span** V when $\mathbf{b}$ is surjective, that is, when every vector $v \in V$ can be written as a linear combination of $b_1, \ldots, b_n$.
3. We say that $(b_1, \ldots, b_n)$ form a **basis** for V when $\mathbf{b} : k^n \xrightarrow{\cong} V$ is a linear isomorphism. In such a case n is called the **dimension** of V and every $v \in V$ can be written in a unique way as a linear combination of the base elements, $v = x_1 b_1 + \cdots + x_n b_n$ and we call $\mathbf{b}^{-1}(v) = (x_1, \ldots, x_n)$ the **coordinates** of v in such a basis.

Lemma 2.1 *Every sequence of linearly independent vectors $v_1, \ldots, v_k$ can be extended to a basis for V.*

Lemma 2.2 *Every spanning sequence of vectors $v_1, \ldots, v_k$ contains a basis for V.*

Lemma 2.3 *If a linear map sends a basis to a basis, then it is an isomorphism.*

Proposition 2.4 *Let $U \subseteq V$ be a linear subspace of a finite dimensional vector space V.*

1. *$\dim U \leq \dim V$, and the equality holds only when $U = V$.*
2. *$\dim(V/U) = \dim V - \dim U$.*

Corollary 2.5 *Let $f : V \longrightarrow W$ be a linear map. Then*

$$\boxed{\dim V = \dim \operatorname{Ker} f + \dim \operatorname{Im} f}$$

Corollary 2.6 *Let $f : V \longrightarrow W$ be a linear map between vector spaces of the same finite dimension. Then the following are equivalent:*

(a) *f is isomorphism,*

(b) *f is injective,*

(c) *f is surjective.*

3 Matrix of a linear map

Let $f : V \longrightarrow V'$ be a linear map and let $(b_1, \ldots b_n)$ and $(b'_1, \ldots b'_m)$ be basis for V and V', so that

$$f(b_j) = a_{1j} b'_1 + \cdots + a_{mj} b'_m$$

for some unique scalars $a_{ij} \in k$, $1 \leq i \leq m$, $1 \leq j \leq n$.

Definition. The matrix $A = (a_{ij})$ is called the **matrix of** f with respect to the basis $(b_1, \ldots b_n)$ and $(b'_1, \ldots b'_m)$. That is, if $\mathbf{b} : k^n \longrightarrow V$ and $\mathbf{b}' : k^m \longrightarrow V'$ are the basis of V and V', then A is the matrix of the linear map $(\mathbf{b}')^{-1} \circ f \circ \mathbf{b} : k^n \longrightarrow k^m$.

Note that A is the matrix whose columns are the coordinates of $f(b_j)$ in the basis $(b'_1, \ldots b'_m)$.

If $v \in V$ has coordinates $X = (x_1, \ldots, x_n)^T$ (column vector) in the basis $(b_1, \ldots b_n)$, then the coordinates X' of $f(v)$ in the basis $(b'_1, \ldots b'_m)$ are given by $X' = AX$.

Proposition 3.1 *Let A be the matrix of a linear map $f : V \longrightarrow V'$ in some basis. Then*

$$\boxed{\dim \operatorname{Im} f = \operatorname{rank} A}$$

and therefore

$$\boxed{\dim \operatorname{Ker} f = \#(\text{columns of } A) - \operatorname{rank} A}$$

Corollary 3.2 *Let $v_1, \ldots, v_r \in V$. Let B be the matrix whose columns are the coordinates of $v_1, \ldots, v_r$ in some basis.*

1. *These vectors are linearly independent if and only if $\operatorname{rank} B = r$. In particular, if $\operatorname{rank} B = r = \dim V$, they form a basis for V; and if $r > \dim V$, then they are linearly dependent.*
2. *These vectors span V if and only if $\operatorname{rank} B = \dim V$.*

Corollary 3.3 *Let A be the matrix of a linear map $f : V \longrightarrow W$ between vector spaces of the same finite dimension. Then f is an isomorphism if and only if $\det A \neq 0$.*

4 Dual vector space

If V, W are vector spaces, the set of linear maps $f : V \longrightarrow W$ is again a vector space denoted by $\operatorname{Hom}(V, W)$, of dimension $\dim \operatorname{Hom}(V, W) = (\dim V)(\dim W)$.

Definition. Let V be a vector space. The **dual vector space** of V is $V^* := \operatorname{Hom}(V, k)$.

Proposition 4.1 *Let $(b_1, \ldots, b_n)$ be a basis for V. There exists a unique basis $(\beta^1, \ldots, \beta^n)$ for V^* with the property that $\beta^i(b_j) = \delta^i_j$. This basis is called the* ***dual basis*** *of $(b_1, \ldots, b_n)$.*

If $\phi \in V^*$, then its coordinates in the dual basis are

$$\boxed{\phi = \phi(b_1)\beta^1 + \ldots + \phi(b_n)\beta^n.}$$

Definition. Let $f : V \longrightarrow W$ be a linear map. The **pull-back** or **dual** of f is the linear map

$$f^* : W^* \longrightarrow V^*\ ,\ f^*(\phi) := \phi \circ f.$$

If $(b_1, \ldots, b_n)$ is a basis for V and $(\beta^1, \ldots, \beta^n)$ is its dual basis, given $\omega \in W^*$ the coordinates of $f^*(\omega)$ are

$$\boxed{f^*(\omega) = \omega(f(b_1))\beta^1 + \ldots + \omega(f(b_n))\beta^n.}$$

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