Differentiable Manifolds 1 Summary

A short overview of important definitions, lemmas and theorems from the course Differentiable Manifolds 1, Leiden University, 2016, given by Roland van der Veen.

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Note: In this summary, all vector spaces are assumed to be finite dimensional.

1 Preliminaries

Definition 1.2.1 (Differentiability) *(a) Let V, W be vector spaces and $A \subseteq V$ an open set and $Y \subseteq W$. A function $f : A \to Y$ is called* **differentiable** *at $a \in A$ if there exists a linear map $\lambda_{f,a} \in L(V, W)$ such that*

$$\lim_{v \to 0} \frac{|f(a+v) - f(a) - \lambda_{f,a}(v)|}{|v|} = 0.$$

We will denote the first derivative $\lambda_{f,a}$ by $\lambda_{f,a} = Df(a)$.

(b) Let V, W be vector spaces and let $X \subseteq V$ and $Y \subseteq W$. A function $f : X \to Y$ is called **differentiable** *($k \in \mathbb{N}$) if there exists an open set $A \subseteq V$ such that $F : A \to Y$ is C^k-differentiable and such that $F\big|_X = f$.*

Lemma 1.2.1 (Chain rule) *Let U, V, W be vector spaces and let $A \subseteq U$, and $B, C \subseteq V$ and let $E \subset W$. Furthermore, let $f : A \to C$ and $g : B \to E$ be differentiable functions. Then we have for all $a \in A$*

$$D(g \circ f)(a) = Dg(f(a)) \circ Df(a).$$

Definition 1.2.2 *Let V, W be vector spaces and $X \subseteq V$ and $Y \subseteq W$. A function $f : X \to Y$ is called* **C^k-differentiable** *($k \in \mathbb{N}$) if all its partial derivatives of order k or less exist and are continuous.*

Definition 1.2.4 (Diffeomorphism) *Let V, W be vector spaces and $X \subseteq V$ and $Y \subseteq W$. A function $f : X \to Y$ is called a* **C^k-diffeomorphism** *($k \in \mathbb{N}$) if f is bijective and if f and f^{-1} are both C^k-differentiable.*

Lemma 1.2.3 *Let V be a vector space and let $X, Y \subseteq V$. Furthermore, let $f : X \to Y$ be a C^1-diffeomorphism. Then for any $x \in X$, the derivative $Df(x)$ is an isomorphism, i.e.* $\det Df(x) \neq 0$.

Theorem 1.2.1 (Inverse function theorem) *Let V be a vector space and let $X, Y \subseteq V$ be open sets. Furthermore, let $f : X \to Y$ be a C^k-differentiable ($k \in \mathbb{N}$) and $x \in X$. If the derivative $Df(x)$ is an isomorphism, then there exist open neighbourhoods $X_0 \subseteq X$ and $Y_0 \subset Y$ such that $x \in X_0$ and $f(x) \in Y_0$ such that $f\big|_{X_0}$ is a C^k-diffeomorphism from X_0 onto Y_0.*

2 Manifolds

Definition 2.1.1 (Coordinate patch) *Let V be a vector space and $A \subseteq V$ an open set and $B \subseteq \mathbb{R}^n$ ($n \in \mathbb{N}$) an open set. Then $\phi : B \to A$ is a* **parametrization** *if ϕ is a C^k-diffeomorphism ($k \in \mathbb{N}$). We then call ϕ^{-1} a* **chart** *and A a* **coordinate patch**.

Definition 2.1.2 (Manifold) *Let V be a vector space and $X \subseteq V$. Then X is a* **C^k-n-dimensional manifold** ($k, n \in \mathbb{N}$) *if there exists C^k-diffeomorphic parametrizations $\phi_i : B_i \to A_i$ ($i \in \mathbb{N}$) with $B_i \subseteq \mathbb{R}^n$ open sets and $A_i \subset X$ coordinate patches such that $X = \bigcup_{i=1}^{\infty} A_i$.*

Theorem 2.1.1 (Rank theorem) *Let V, W be vector spaces and let $X \subseteq V$ be an open set and let $Y \subset W$. Furthermore, let $f : X \to Y$ be C^k-differentiable ($k \in \mathbb{N}$). If for all $x \in X$ the derivative $Df(x)$ has constant rank $r \in \mathbb{N}$, then for all $w \in W$ with $f^{-1}(w) \neq \emptyset$, the inverse image $f^{-1}(w) \subset X$ is a C^k-n-dimensional manifold, where $n = \dim(V) - r$.*

Definition 2.2.1 (Tangent space) *Let V be a vector space and $X \subseteq V$ a C^k-n-dimensional manifold ($k, n \in \mathbb{N}$). Furthermore, let $a \in \mathbb{R}^n$ and $x \in X$, and let $\phi : B \to A$ with $B \subseteq \mathbb{R}^n$ an open set and $A \subset V$ a coordinate patch be a parametrization such that $\phi(a) = x$. Then the* **tangent space** *of X at x is $T_x X = \mathrm{im} D\phi(a)$.*

Lemma 2.2.1 *Let V be a vector space and $X \subseteq V$ a C^k-n-dimensional manifold ($k, n \in \mathbb{N}$), and let $x \in X$. Then the dimension of $T_x X$ equals the dimension of X.*

Definition 2.2.2 (Derivative of manifolds) *Let V, W be vector spaces and let $X \subseteq V$ and $Y \subseteq W$ be a C^{k_1}-n_1-dimensional manifold and a C^{k_2}-n_2-dimensional manifold respectively ($k_1, k_2, n_1, n_2 \in \mathbb{N}$). Furthermore, let $f : X \to Y$ be C^1-differentiable. Then the* **derivative** *is defined as $Df(x) = DF(x)\big|_{T_x X}$, where $F : A \to Y$ is a C^1-differentiable such that $F\big|_X = f$ and where $A \subseteq V$ is an open set such that $x \in A$.*

Definition 2.3.1 (Geodesic) *Let V be a vector space and $X \subseteq V$ a C^2-n-dimensional manifold ($n \in \mathbb{N}$). A* **geodesic** *is a C^2-differentiable map $\gamma : [a,b] \to X$ (where $a, b \in \mathbb{R}$ such that $a < b$) such that $\ddot{\gamma}(t) \perp T_{\gamma(t)}X$ for all $t \in [a,b]$, with $\ddot{\gamma}(t) = D_1(D_1\gamma(t))$.*

Theorem 2.3.1 (Existence and uniqueness of geodesics) *Let V be a vector space and $X \subseteq V$ a C^2-n-dimensional manifold ($n \in \mathbb{N}$). Then for any point $x \in X$ and $v \in T_xX$, there exists a unique geodesic $\gamma : [0,1] \to X$ with $\gamma(0) = x$ and $\dot{\gamma}(0) = v$.*

Definition 2.4.1 (Vector field) *Let V be a vector space and $X \subseteq V$ a C^k-n-dimensional manifold ($k, n \in \mathbb{N}$). A* **vector field** *is a C^l-differentiable map $F : X \to V$ ($l \in \mathbb{N}$). A* **tangent vector field** *is a C^l-differentiable map $F : X \to V$ such that for all $x \in X$, we have $F(x) \in T_xX$.*

Definition 2.4.2 (Integral curve) *Let V be a vector space and $X \subseteq V$ a C^k-n-dimensional manifold ($k, n \in \mathbb{N}$). An* **integral curve** *is a C^1-differentiable map $\gamma : [a,b] \to X$ (where $a, b \in \mathbb{R}$ such that $a < b$) for the vector field $F : X \to V$ if we have such that $D\gamma(t)(e_1) = F(\gamma(t))$ for all $t \in [a,b]$, where e_1 is the first standard basis vector of $\mathbb{R}^n$.*

Definition 2.4.2 *Let V be a vector space and $X \subseteq V$ a C^k-n-dimensional manifold ($k, n \in \mathbb{N}$), and $F : X \to V$ a C^1-differentiable vector field. Then for all $x \in X$, there exists an integral curve $\gamma : [0,1] \to X$ such that $\gamma(0) = x$.*

Lemma 2.5.1 (Normal field) *Let V be an $n+1$-dimensional vector space and $X \subseteq V$ an n-dimensional hyperspace ($n \in \mathbb{N}$). Then $\mathcal{N} : X \to \mathbb{S}^n \subset \mathbb{R}^{n+1}$ is a* **smooth normal field** *if for all $x \in X$, there exists an open neighbourhood $U \subset X$ such that $\mathcal{N}(x) \perp T_xX$ for all $x \in U$. Such a normal field always exists.*

Definition 2.5.1 (Gaussian curvature) *Let V be a $n+1$-dimensional vector space and $X \subseteq V$ a n-dimensional hyperspace ($n \in \mathbb{N}$) and $\mathcal{N} : X \to \mathbb{S}^n \subset \mathbb{R}^{n+1}$ a smooth normal field. Then the* **Gaussian curvature** *at a point $x \in X$ is defined as $\kappa(x) = \det D\mathcal{N}(x)$, where $D\mathcal{N}(x) : T_xX \to \mathbb{R}^{n+1}\big|_{T_xX}$ is known as the shape operator.*

Definition 2.6.1 (a) *Let V be a vector space and $k \in \mathbb{N}$. A map $T : V^k \to \mathbb{R}$ is called* k**-linear** *if its linear in each of its k arguments. The set of all such maps is called $Mult^k(V)$. By convention, $Mult^0(V) = \mathbb{R}$.*

(b) *Let V be a vector space and $k \in \mathbb{N}$. A map $T : V^k \to \mathbb{R}$ is called* **alternating** *if it is k-linear and if the sign changes when two arguments are interchanged. The set of all such maps is called $Alt^k(V) \subset Mult^k(V)$. By convention, $Alt^0(V) = \mathbb{R}$.*

Definition 2.6.2, 2.7.3 (Pullback) *(a) Let V, W be vector spaces and $k \in \mathbb{N}$, and let $f : V \to W$ be a linear map. The* **pullback** *is the map $f^* : Mult^k(W) \to Mult^k(V)$ defined by $f^*(T)(v_1, \ldots, v_k) = T(f(v_1), \ldots, f(v_k))$, where $T \in Mult^k(W)$ and $v_1, \ldots, v_n \in V$.*

(b) Let V, W be vector spaces and $X \subset V$ and $Y \subset W$ manifolds, and let $k \in \mathbb{N}$, and let $f : X \to Y$ be C^l-differentiable ($l \in \mathbb{N}$). The **pullback** *is the map $f^* : \Omega^k(Y) \to \Omega^k(X)$ defined by $f^*(\omega)(x) = (Df(x))^*(\omega(f(x)))$, where $\omega \in \Omega^k(Y)$ and $x \in X$. The definition of $\Omega^k(Y)$ and $\Omega^k(X)$ will follow in Definition 2.7.1.*

Lemma 2.6.1 (Determinant lemma) *Let V be an n-dimensional vector space ($n \in \mathbb{N}$). Then $\dim Alt^n(V) = 1$, and for all $B \in L(V, V)$ and $T \in Alt^n(V)$, we have that $B^*T = (\det B)T$.*

Definition 2.6.3 (Tensor product) *Let V be a vector space and let $S \in Mult^m(V)$ and $T \in Mult^n(V)$, ($m, n \in \mathbb{N}$). Then $S \otimes T \in Mult^{m+n}(V)$ is indeed multilinear if defined as $(S \otimes T)(v_1, \ldots, v_m, v_{m+1}, \ldots, v_{m+n}) = S(v_1, \ldots, v_m)T(v_{m+1}, \ldots, v_{m+n})$.*

Definition 2.6.4.0 (Alt) *Let V be a vector space and $k \in \mathbb{N}$. Then we define $\mathrm{Alt} : Mult^k(V) \to Alt^k(V)$ by $\mathrm{Alt}(T)(v_1, \ldots, v_k) = \frac{1}{k!} \sum_{\pi \in S_k} \mathrm{sgn}(\pi) T(\pi(v_1), \ldots, \pi(v_k))$, where $v_1, \ldots, v_k \in V$.*

Definition 2.6.4 (Wedge product) *Let V be a vector space and $k \in \mathbb{N}$ and let $S \in Mult^m(V)$ and $T \in Mult^n(V)$, ($m, n \in \mathbb{N}$). Then $S \wedge T \in Alt^{m+n}(V)$ is indeed alternating if defined as $(S \wedge T) = \binom{m+n}{m} \mathrm{Alt}(T \otimes S)$.*

Lemma 2.6.2, 2.6.6 *Let V be an n-dimensional vector space ($n \in \mathbb{N}$), and let $\{b_1^*, \ldots, b_n^*\}$ be the dual basis to the basis $\{b_1, \ldots, b_n\}$ of V.*

(a) Then the set of k tensor products of elements of the (above) dual basis of V forms a basis for $Mult^k(V)$ ($k \in \mathbb{N}$), and $\dim Mult^k(V) = n^k$.

(b) Then the set of k wedge products of different elements of the (above) dual basis of V forms a basis for $Alt^k(V)$ ($k \in \mathbb{N}$), and $\dim Alt^k(V) = \binom{n}{k}$.

Definition 2.7.1 (Tensor and differential form) *(a) Let V be a vector space and $X \subseteq V$ a C^m-n-dimensional manifold ($m, n \in \mathbb{N}$). A* **C^l-class k-tensor** *is a C^l-differentiable map ($l \in \mathbb{N}$) $\omega : X \to Mult^k(V)$ ($k \in \mathbb{N}$) such that $\omega(x)\big|_{(T_xX)^k} \in Mult^k(T_xX)$.*

(b) *Let V be a vector space and $X \subseteq V$ a C^m-n-dimensional manifold $(m, n \in \mathbb{N})$. A C^l-**differential k-form** is a C^l-differentiable map $(l \in \mathbb{N})$ $\omega : X \to Mult^k(V)$ $(k \in \mathbb{N})$ such that $\omega(x)\big|_{(T_xX)^k} \in Alt^k(T_xX)$. The set of all differential k-forms on X is called $\Omega^k(X)$.*

Definition 2.7.2 (Orientation) *(a) Let V be a vector space and $X \subseteq V$ a C^m-n-dimensional manifold $(m, n \in \mathbb{N})$. An **orientation** is a C^1-differential n-form such that for all $x \in X$, we have that $\omega(x) \neq 0$. A manifold together with a choice of orientation is a called an oriented manifold.*

(b) *Let V be a vector space and $X \subseteq V$ a C^m-n-dimensional manifold $(m, n \in \mathbb{N})$. Two orientations ω, ω' are said to be equivalent if there exists a function $f : X \to \mathbb{R}$ such that, for all $x \in X$, we have $f(x) > 0$ and $\omega(x) = f(x)\omega'(x)$.*

(c) *Let V, W be vector spaces and let $X \subseteq V$ and $Y \subseteq W$ be a C^{k_1}-n_1-dimensional oriented manifold (with orientation $\omega_X \in \Omega^k(X)$) and a C^{k_2}-n_2-dimensional manifold (with orientation $\omega_Y \in \Omega^k(Y)$) respectively $(k_1, k_2, n_1, n_2 \in \mathbb{N})$. A map $f : X \to Y$ is called **orientation preserving** if $f^*(w_Y)$ is equivalent to ω_X.*

Definition 2.7.4 (Exterior derivative) *Let V be a vector space and $X \subseteq V$ a C^m-n-dimensional manifold $(m, n \in \mathbb{N})$. An **exterior derivative** is a map $\Omega^k(X) \to \Omega^{k+1}(X)$ $(k \in \mathbb{N})$ defined by $d\omega = \mathrm{Alt}(D\omega)$.*

Lemma 2.7.1 *Let V be an n-dimensional vector space $(n \in \mathbb{N})$ and $U \subset \mathbb{R}^{n+1}$ an open set. Then all differential k-forms $(k \in \mathbb{N})$ can be written as k wedge products of products of a function from $U \to \mathbb{R}$ and dx^i for certain values of $i \in \{1, \dots, n\}$, where $x^i : X \to \mathbb{R}$ is the projection on the i-th coordinate.*

Lemma 2.7.2 *Let V, W be vector spaces and let $X \subseteq V$ and $Y \subseteq W$ be a C^{k_1}-n_1-dimensional manifold and a C^{k_2}-n_2-dimensional manifold respectively $(k_1, k_2, n_1, n_2 \in \mathbb{N})$.*

1. *For any $\omega \in \Omega^k(X)$ $(k \in \mathbb{N})$, we have that $d(d\omega) = 0$.*
2. *For any differentiable function $f : X \to Y$ and for any $\omega \in \Omega^k(X)$ $(k \in \mathbb{N})$, we have that $f^*(d\omega) = d(f^*(\omega))$.*
3. *For any $\alpha \in \Omega^k(X)$ $(k \in \mathbb{N})$ and $\beta \in \Omega^l(X)$ $(l \in \mathbb{N})$, we have that $d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^k \alpha \wedge d\beta$.*

3 Integration

Definition 3.1.1 (Support) *Let V, W be vector spaces and $X \subseteq V$ a C^m-n-dimensional manifold ($m, n \in \mathbb{N}$). The* **support** *of a function $f : X \to V$ is the closure* $\operatorname{supp}(w) = \overline{\{x \in X \mid f(x) \neq 0\}}$.
The set of all differential k-forms ($k \in \mathbb{N}$) with compact support (i.e. where the support is a compact subset of V) is denoted by $\Omega_c^k(X)$.

Lemma 3.1.1 *Let $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^n$ ($n \in \mathbb{N}$) be open set, and let $f : X \to Y$ be an orientation preserving diffeomorphism. Furthermore, let $\omega \in \Omega_c^n(Y)$. Then we have that $\displaystyle\int_Y \omega = \int_X f^*(\omega)$.*

Definition 3.2.1 (Half space) *Let W be a vector space. Furthermore, let $\alpha \in W^*$. Then the halfspace H_α of W is defined as $H_\alpha = \alpha^{-1}[0, \infty)$. The boundary of H_α is $\partial H_\alpha = \ker \alpha = \alpha^{-1}\{0\}$. An element of $w \in W$ is said to be outward pointing if $\alpha(w) < 0$.*

Definition 3.2.2 (Coordinate patch with boundary) *Let V be a vector space and $A \subseteq V$ an open set and let $\alpha \in (\mathbb{R}^n)^*$ ($n \in \mathbb{N}$). Furthermore, let $B \subseteq H_\alpha$ an open set. Then $\phi : B \to A$ is a* **parametrization** *if ϕ is a C^k-diffeomorphism ($k \in \mathbb{N}$). We then call ϕ^{-1} a* **chart** *and A a* **coordinate patch with boundary**. *By definition, the boundary of B is $\partial B = B \cap \partial H_\alpha$.*

Definition 3.2.3 (Manifold with boundary) *Let V be a vector space and $X \subseteq V$, and let $\alpha \in (\mathbb{R}^n)^*$ ($n \in \mathbb{N}$). Then X is a* **C^k-n-dimensional manifold with boundary** *($k \in \mathbb{N}$) if there exists C^k-diffeomorphic parametrizations $\phi_i : B_i \to A_i$ ($i \in \mathbb{N}$) with $B_i \subseteq H_\alpha$ open sets and $A_i \subset X$ coordinate patches with boundary such that $X = \displaystyle\bigcup_{i=1}^{\infty} A_i$. Then, the boundary of X is $\partial X = \displaystyle\bigcup_{i=1}^{\infty} \phi_i(\partial B_i)$.*

Theorem 3.3.1 (Stokes) *Let V be a vector space and $X \subseteq V$ a C^m-n-dimensional oriented manifold with boundary ($m, n \in \mathbb{N}$). Furthermore, let $\omega \in \Omega_c^n(X)$. Then we have that $\displaystyle\int_X d\omega = \int_{\partial X} \omega$.*

Definition 3.4.0 *(a) Let V be a vector space and $X \subseteq V$ a C^m-n-dimensional manifold ($m, n \in \mathbb{N}$). Then X is connected if X is connected as topological subspace of V.*

(b) Let V be a vector space and $X \subseteq V$ a C^m-n-dimensional manifold ($m, n \in \mathbb{N}$). Then X is compact if X is compact as topological subspace of V.

Theorem 3.4.1 (Morera) *Let V be a vector space and $X \subseteq V$ a C^m-n-dimensional oriented connected manifold with boundary ($m, n \in \mathbb{N}$). Furthermore, let $\alpha \in \Omega_c^n(X)$. Then there exists a $\omega \in \Omega_c^{n-1}(X)$ such that $d\omega = \alpha$ if and only if $\int_X \alpha = 0$.*

Definition 3.4.1 (Brouwer degree) *Let V, W be vector spaces and let $X \subseteq V$ and $Y \subseteq W$ be a C^{k_1}-n-dimensional compact oriented connected manifold and a C^{k_2}-n-dimensional compact oriented connected manifold respectively ($k_1, k_2, n \in \mathbb{N}$). Furthermore, let $f : X \to Y$ be a map. Then the* **Brouwer degree** $\mathrm{deg(f)}$ *is the unique number (in $\mathbb{R}$) satisfying $\int_X f^*(\omega) = \mathrm{deg(f)} \int_Y \omega$ for all $\omega \in \Omega^n(Y)$.*

Lemma 3.4.1 (Homotopic invariance of the Brouwer degree) *Let V, W be vector spaces and let $X \subseteq V$ and $Y \subseteq W$ be a C^{k_1}-n_1-dimensional compact oriented connected manifold and a C^{k_2}-n_2-dimensional compact oriented connected manifold respectively ($k_1, k_2, n_1, n_2 \in \mathbb{N}$). Furthermore, let $f : X \to Y$ and $g : X \to Y$ be C^1-differentiable. If f and g are homotopic, then* $\mathrm{deg(f)} = \mathrm{deg(g)}$.

Definition 3.4.4 (Index of a vector field) *Let V be an inner product vector space and $X \subseteq V$ a C^m-n-dimensional manifold ($m, n \in \mathbb{N}$). Furthermore, let $f : X \to V$ be a vector field, and let $z \in X$ such that there exists a neighbourhood $U \subset X$ with $z \in U$ such that $f(u) \neq 0$ for all $u \in U \setminus \{z\}$, and such that U is a C^{k_1}-n-dimensional manifold with boundary such that ∂U is diffeomorphic to $\mathbb{S}^{n-1}$. Then the* **index** *of f at z is defined to be $\mathrm{Index}(F; z) = \mathrm{deg}\tilde{f}$, where $\tilde{f} : \partial U \to S^{n-1}$ is defined by $\tilde{f}(x) = \dfrac{f(x)}{|f(x)|}$.*

Theorem 3.5.1 (Gauss-Bonnet) *Let $V = \mathbb{R}^{2n+1}$ be an inner product vector space and $X \subseteq V$ a C^m-$2n$-dimensional compact oriented connected manifold ($m, n \in \mathbb{N}$) with empty boundary. Furthermore, let $f : X \to V$ be a C^1-differentiable tangent field such that for all $z \in X$ with $f(z) = 0$, there exists a neighbourhood $U \subset X$ with $z \in U$ such that $f(u) \neq 0$ for all $u \in U \setminus \{z\}$. Then*

$$\int_x \kappa \mathrm{Vol}_X = \frac{Volume(S^{2n})}{2} \sum_{z:F(z)=0} \mathrm{Index}(F; z),$$

where $\mathrm{Vol}_X \in \Omega^{2n}(X)$ is the unique element such that for all $x \in X$ any orthonormal basis $\{b_1, \ldots, b_{2n}\}$ of $T_x X$ for which all the basis vectors are inward pointing, we have that $\mathrm{Vol}_X(x)(b_1, \ldots, b_{2n}) = 1$.

References

[1] Roland van der Veen, Differentiable Manifolds, 2016