Exercises Chapter 3 Manifolds 1 2018-2019
Version 2

1. Prove for a vector field $F : \mathbb{R}^3 \to \mathbb{R}^3$ and a smooth map $f : \mathbb{R}^3 \to \mathbb{R}$ that $\mathrm{div}(\mathrm{curl}(F)) = 0$ and that $\mathrm{curl}(\mathrm{grad}(f)) = 0$

   (a) by explicit calculation, or

   (b) by using the integral theorems in the beginning of chapter 3.

2. Exact sequences

   (a) Let $\boldsymbol{T}_0^0\mathbb{R}^3$ be the set of all smooth ($\mathbb{R}$-valued) functions on $\mathbb{R}^3$ and $\boldsymbol{T}_0^1\mathbb{R}^3$ be the set of all smooth vector fields on $\mathbb{R}^3$. Prove that the following is an exact sequence:

   $$\boldsymbol{T}_0^0\mathbb{R}^3 \overset{grad}{\to} \boldsymbol{T}_0^1\mathbb{R}^3 \overset{curl}{\to} \boldsymbol{T}_0^1\mathbb{R}^3 \overset{div}{\to} \boldsymbol{T}_0^0\mathbb{R}^3$$

   (This means that the image of a map in the sequence equals the kernel of the next map in the sequence.)

   (b) Let $\boldsymbol{T}_1^0\mathbb{R}^3$ be the set of all smooth covector fields on $\mathbb{R}^3$, meaning that for $\omega \in \boldsymbol{T}_1^0\mathbb{R}^3$ it holds $\omega \colon \mathbb{R}^3 \to (\mathbb{R}^3)^*$. Show that the gradient, curl and divergence induce maps $grad_*$, $curl_*$ and div_* such that the following is an exact sequence:

   $$\boldsymbol{T}_0^0\mathbb{R}^3 \overset{grad_*}{\to} \boldsymbol{T}_1^0\mathbb{R}^3 \overset{curl_*}{\to} \boldsymbol{T}_1^0\mathbb{R}^3 \overset{div_*}{\to} \boldsymbol{T}_0^0\mathbb{R}^3$$

3. The Maxwell equations describing classical electrodynamics are:

   $$\vec{\nabla} \cdot \vec{E} = 4\pi\rho \qquad \vec{\nabla} \cdot \vec{B} = 0$$
   $$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \vec{\nabla} \times \vec{B} = 4\pi\vec{J} + \frac{\partial \vec{E}}{\partial t}$$

   Here $\vec{E}$ and $\vec{B}$ are $\mathcal{C}^1$ vector fields on $\mathbb{R}^3$ (space) that change in jointly $\mathcal{C}^1$ fashion depending on some parameter $t \in \mathbb{R}$ (time). $\vec{E}$ is called the electrical field and $\vec{B}$ the magnetic field. Moreover ρ is an ($\mathbb{R}$-valued) function on $\mathbb{R}^3$ called the charge distribution and $\vec{J}$ is a vector field on $\mathbb{R}^3$ called the current distribution both also dependent on time t.

   (a) By introducing two new variables $\vec{A}$ - a vector field - and ϕ - a function -, show that the four Maxwell equations can be reduced to two equations.

   (b) What kind of freedom is there involved in choosing $\vec{A}$ and ϕ?

   (c) For what choice of $\vec{A}$ and ϕ do the two Maxwell equations become symmetric (in appearance on paper) to each other? What do the equations then look like?

4. Calculate $\int_{[0,1]} x^3$ via our definition.

5. Can we also use our definition of integrals to calculate improper integrals? If yes, calculate $\int_{[1,\infty)} \frac{1}{x}$, if no, explain why not.

6. Generalize our definition of integrals such that for open sets $P \subset \mathbb{R}^n$ and continuous functions $f : P \to \mathbb{R}$, we have that
$$\int_P f = \int_{\overline{P}} f.$$

7. Let R, S be compact subsets of $\mathbb{R}$, and $f : R \cup S \to \mathbb{R}$ continuous. Prove:
$$\int_{R\cup S} f + \int_{R\cap S} f = \int_R f + \int_S f.$$

8. Let $D \subset \mathbb{R}^2$ be compact and path connected, and let $f : D \to \mathbb{R}$ be continuous. Prove that there exists a point $(x_0, y_0) \in D$ such that
$$\int_D f = f(x_0, y_0) \cdot (\text{area of } D).$$

9. Another possible theory of integration could be the following: let $R \subset \mathbb{R}^n$ be compact, and $f : R \to \mathbb{R}$ continuous. Then choose a chain $Q_1 \subsetneq Q_2 \subsetneq \cdots$ such that $Q_m \subseteq R \cap \mathbb{Q}^n$ and that $0 < |Q_m| < \infty$ for all $m \in \mathbb{N}$. Then define
$$\int_R f = \lim_{n\to\infty} I_{R,n}(f),$$
where $I_{R,n}(f) = \frac{1}{|Q_n|} \sum_{q\in Q_n} f(q)$.
Is this a correct definition for an integral? If yes, prove that this definition is equal to our old definition. If no, give a counterexample and a possible fix to this definition.

10. Prove that our definition of integrals coincides with the definition of the Riemann integral for continuous functions from closed sets of $\mathbb{R}^n$ to $\mathbb{R}$.

11. (Lemma 11, part 2) Prove that $C, \tilde{C} \in L(V, \mathbb{R}^k)$ are equivalent if and only if either both kernels have codimension $< k$ or the kernels agree and $\tilde{C} = GC$ for some $G \in L(\mathbb{R}^k, \mathbb{R}^k)$ with $\det G = 1$.

12. Recall from the previous exercise sheet the following construction. Let V be a real vector space. Then we define $V \wedge V := (V \times V)/U$, where $V \times V := \operatorname{span}\{(v, w) \mid v, w \in V\}$ and
$$U = \operatorname{span}\left\{ \begin{matrix} (\lambda \cdot v, w) - (v, \lambda \cdot w) \\ (v + v', w) - (v, w) - (v', w) \\ (v, w + w') - (v, w) - (v, w') \\ (v, v) \end{matrix} \;\middle|\; v, v', w, w' \in V, \lambda \in \mathbb{R} \right\}.$$
Prove that $V \wedge V$ is isomorphic to $\Lambda^2 V$ as vector spaces.

13. Let V be a real vector space. Suppose that $\Lambda^k(V^*)$ and $\Lambda^k V$ are finite dimensional. Prove that $\Lambda^k(V^*)$ and $(\Lambda^k V)^*$ are isomorphic as vector spaces.

14. Let V, W be real vector spaces, and let $H \in L(V, W)$. Prove or disprove the following statements:

    (a) If H is injective, then H^* is surjective.

    (b) If H is surjective, then H^* is injective.

    (c) If H is an isomorphism, then H^* is an isomorphism.

15. Let V, W be real vector spaces. One may push-forward a k-vector field on W to a k-vector field on V using formulas similar to those of the pull-back. Can you make this precise?

16. Prove integration by parts, i.e. prove that if $f, g\colon [a,b] \to \mathbb{R}$ are $\mathcal{C}^1$ then the following holds:

$$\int_a^b f(x)g'(x)dx = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x)dx$$

17. The mean value theorem of integration is as follows: if $f\colon [a,b] \to \mathbb{R}$ is continuous on $[a,b]$, then there exists a $c \in (a,b)$ such that:

$$\int_a^b f(x)dx = f(c)(b-a)$$

    (a) Prove that the mean value theorem of integration is equivalent to the following statement: if $f\colon [a,b] \to \mathbb{R}$ is $\mathcal{C}^1$, then there exists a $c \in (a,b)$ such that:

$$f(b) - f(a) = f'(c)(b-a)$$

    (b) Show that the mean value theorem of integration is not equivalent to the mean value theorem of differentiation, i.e. (Theorem 2.2 on page 14) if $f\colon [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b), then there exists a $c \in (a,b)$ such that:

$$f(b) - f(a) = f'(c)(b-a)$$

18. Let $(f_n\colon [0,a] \to \mathbb{R})_{n\in\mathbb{N}}$ be a family of $\mathcal{C}^1$ functions such that $f_n([0,a]) \subset [0,1]$, $f_n([\frac{1}{n}, a]) = \{1\}$ and $f_n(0) = 0$ for all n.

    (a) Let $\phi\colon [0,a] \to \mathbb{R}$ be a $\mathcal{C}^1$ function. Prove that:

$$\lim_{n\to\infty} \int_0^a f_n'(x)\phi(x)dx = \phi(0)$$

(b) Show that if ϕ is just continuous on $[0, a]$, then the equation above does not hold in general.

19. (Defining $\Lambda^k V$ via Alt.) Let V, W be finite dimensional real vector spaces. Then we define $V \otimes W := (V \times W)/U$, where $V \times W := \operatorname{span}\{(v, w) \mid v \in V, w \in W\}$ and

$$U = \operatorname{span}\left\{ \begin{array}{c} (\lambda \cdot v, w) - (v, \lambda \cdot w) \\ \lambda \cdot (v, w) - (\lambda \cdot v, w) \\ (v + v', w) - (v, w) - (v', w) \\ (v, w + w') - (v, w) - (v, w') \end{array} \middle| \; v, v' \in V, w, w' \in W, \lambda \in \mathbb{R} \right\}.$$

For $k > 2$, define $V^{\otimes k} := \overbrace{V \otimes \cdots \otimes V}^{k \text{ times}}$ inductively. We write $v \otimes w \in V \otimes W$ for $v \in V$, $w \in W$.

(a) Find a basis (or the dimension) of $V \otimes W$.

(b) Let $\varphi : V^{\otimes k} \to W$ be a linear map. Show that we can also obtain φ as a multilinear map from V^k to W.

For $k > 0$, define

$$\begin{aligned} \operatorname{Alt} : V^{\otimes k} &\to V^{\otimes k} \\ v_1 \otimes \cdots \otimes v_k &\mapsto \frac{1}{k!} \sum_{\sigma \in S_k} (-1)^\sigma v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}. \end{aligned}$$

By construction, Alt is linear.

(c) Prove that Alt is an alternating map. (Hence the name.)

(d) Is Alt surjective?

(e) Prove that the linear map

$$\begin{aligned} (V^{\otimes k})^* &\to (V^*)^{\otimes k} \\ (v_1 \otimes \cdots \otimes v_k)^* &\mapsto (v_1^* \otimes \cdots \otimes v_k^*) \end{aligned}$$

is an isomorphism, where $\cdot^* : V \to V^*$ is an isomorphism of your choice.

(f) Prove that

$$\ker \operatorname{Alt} = \{v \in V^{\otimes k} \mid \text{for all } w \in (V^*)^{\otimes k} : \; \operatorname{Alt}(w)(\operatorname{Alt}(v)) = 0\}.$$

(g) Define $V^{\times k} = \overbrace{V \times \cdots \times V}^{k \text{ times}}$ inductively. Prove that the map

$$\begin{aligned} \operatorname{Mat} : V^{\times k} &\to L(\mathbb{R}^k, V) \\ (v_1, \ldots, v_k) &\to \begin{pmatrix} | & & | \\ v_1 & \cdots & v_k \\ | & & | \end{pmatrix} \end{aligned}$$

is an isomorphism.[1]

Similarly, there is an isomorphism $\mathrm{Mat}^* : (V^*)^{\times k} \to L(V, \mathbb{R}^k)$.

(h) Prove that there exists a $\lambda = \lambda(V, k) \in \mathbb{R}$ such that

$$\lambda \mathrm{Alt}(w)(\mathrm{Alt}(v)) = \mathcal{I}(\mathrm{Mat}(v), \mathrm{Mat}^*(w))$$

for $v \in V^{\otimes k}$, $w \in (V^*)^{\otimes k}$.

Define $K = \{v \in V^{\times k} \mid \text{for all } w \in (V^*)^{\times k} : \ \mathcal{I}(\mathrm{Mat}(v), \mathrm{Mat}^*(w)) = 0\}$, and let $U^{(k)} \subset V^{\times k}$ the subspace such that $V^{\otimes k} = V^{\times k}/U^{(k)}$.

(i) Prove that $K/U^{(k)} \cong \ker \mathrm{Alt}$.

(j) Prove that $\mathrm{Alt}(V^{\otimes k}) \cong \Lambda^k V$. (Hint: use the third isomorphism theorem.)

Hence we can identify $\mathrm{Alt}(v)$ and $v_1 \wedge \cdots \wedge v_k$ for $v = v_1 \otimes \cdots \otimes v_k \in V^{\otimes k}$.

20. (Connection with multilinear alternating functions)
In the the literature one usually describes k-covectors using alternating multilinear maps. A function $A : V^k \to \mathbb{R}$ is alternating if $A(v_1, \ldots v_k) = 0$ whenever $v_i = v_j$ for some i, j and multilinear means linear in each component. Let $\mathrm{Alt}^k(V)$ be the space of all alternating multilinear maps on V. In this exercise we prove that $\mathrm{Alt}^k(V) \cong \Lambda^k V^*$.

    (a) For any $C \in L(V, \mathbb{R}^k)$ show that the function A_C defined by $A_C(v_1 \ldots v_k) = \mathcal{I}(\wedge_i v_i, C)$ is in $\mathrm{Alt}^k V$.

    (b) Prove that $\mathrm{Alt}^k V$ is a vector space with respect to point-wise linear combinations. Show that the previous part gives a linear map $\Lambda^k V^* \to \mathrm{Alt}^k(V)$.

    (c) Show that the above map is an isomorphism.

21. Let $P \subset \mathbb{R}^n$, $Q \subset \mathbb{R}^m$, $R \subset \mathbb{R}^l$ be open subsets ($l, m, n \in \mathbb{N}$), and let $\varphi : P \to Q$, $\psi : Q \to R$ be smooth maps. Furthermore, let $\omega_1, \omega_2 \in \Omega^1(Q)$ and let $f : Q \to \mathbb{R}^j$ smooth ($j \in \mathbb{N}$).

    (a) Prove that $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$.

    (b) Prove that $d\varphi^*(f) = \varphi^*(df)$.

    (c) Prove that $\varphi^*(\omega_1 \wedge \omega_2) = \varphi^*(\omega_1) \wedge \varphi^*(\omega_2)$.

22. Denote ${S^1}^+ = \{(x, y) \in S^1 \mid x, y \geq 0\}$.

    (a) Calculate $\int_{{S^1}^+ \times S^1} dx^1 \wedge dx^3$.

    (b) Calculate $\int_{\{(1,0),(0,1)\} \times S^1} x^1 dx^3$.

    (c) Can you find a relation between ${S^1}^+ \times S^1$ and $\{(1, 0), (0, 1)\} \times S^1$?

[1] We do not use the pointwise addition in $L(\mathbb{R}^k, V)$ and $L(V, R^k)$!

23. Prove that a k-covector field $f = \sum_{1\le j_1<\cdots<j_k\le n} f_{j_1\cdots j_k} dx^{j_1} \wedge \cdots \wedge dx^{j_k}$ ($k \in \mathbb{N}$) from an open set $P \subset \mathbb{R}^n$ ($n \in \mathbb{N}$) to $\Lambda^k \mathbb{R}^{n*}$ is C^m if and only if all the $f_{j_1\cdots j_k}$ are C^m.

24. (a) Prove that for $k \in \mathbb{N}$, we have that $\partial\partial I^k = 0$. (You may take $k = 2$ if you like.)

    (b) Prove for all chains γ, we have that $\partial\partial\gamma = 0$.

25. Generalize (if possible) our definition of differentiability in Chapter 2 to V and prove that Theorem 1 of Chapter 2 holds in the case that

    (a) V is a finite dimensional real vector space.

    (b) V is an arbitrary vector space.

26. Let $X \subset \mathbb{R}^n$ open and connected ($n \in \mathbb{N}$), and denote $\Omega^n_0 X := \{\omega \in \Omega^n X \mid \omega(p) \neq 0$ for all $p \in X\}$. Define a relation $\sim$ on $\Omega^n_0 X$ by $\omega \sim \mu \Leftrightarrow$ there exists a differentiable function $f_{\omega,\mu} : X \to \mathbb{R}$ such that $\omega = f_{\omega,\mu} \cdot \mu$.

    (a) Prove that $\sim$ is an equivalence relation, and that $f_{\omega,\mu}$ is unique for given ω and μ, and that it is non-zero on all of X.

    (b) Prove that $\Omega^n_0 X/\sim$ is either empty, or that it has two elements. In the latter case, we call X orientable. We call an element of $\Omega^n_0 X/\sim$ an orientation of X.

    (c) Prove that X is orientable, and give an (explicit) orientation for X.

    (d) Prove that the Möbius strip is not orientable.

27. For $n \in \mathbb{N}$, calculate $H_n(\{*\})$.

28. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be defined by $f(x, y, z) = x^2 + y^4 + z^2$, and let $U = f^{-1}(\{1\})$. Calculate $\int_U 2x dy \wedge dz + y dx \wedge dz$.

29. Prove the three integral theorems from analysis at the beginning of Chapter 3.

30. (Closed and exact forms) Let $X \subset \mathbb{R}^n$ be open ($n \in \mathbb{N}$). We say that a k-form $\alpha \in \Omega^k(X)$ is exact if there exists a $\omega \in \Omega^{k-1}(X)$ such that $d\omega = \alpha$. A k-form $\alpha \in \Omega^k(X)$ is closed if $d\alpha = 0$.

    (a) Prove that every exact form is closed.

    (b) Find a closed form that is not exact.

    (c) Let $x^1, \dots, x^n$ be a basis of $\mathbb{R}^n$ ($n \in \mathbb{N}$). Prove that $dx^1 \wedge \cdots \wedge dx^n \in \Omega^n(\mathbb{R}^n)$ is exact.

31. (Morera's Theorem)

    (a) Let $X \subset \mathbb{R}^2$ be open such that ∂X is connected. Prove that $\alpha \in \Omega^1(\partial X)$ is exact if and only if $\int_{\partial X} \alpha = 0$.

(b) Let $X \subset \mathbb{R}^n$ be open and connected. Prove that $\alpha \in \Omega^n(X)$ is exact if and only if $\int_{\partial X} \alpha = 0$.

32. Let V be an n-dimensional vector space over $\mathbb{R}$ ($n < \infty$.)

 (a) Let $\mathcal{B}$ be the set of ordered bases of V. We define an equivalence relation on $\mathcal{B}$ as follows. For $a, b \in \mathcal{B}$ there exists a unique linear map $P\colon V \to V$ such that $b_i = Pa_i$ for all $1 \leq i \leq n$, where a_i is the i-th vector in a etc. We let $a \sim b$ if $\det(P) > 0$. Show that this is an equivalence relation and that there are 2 equivalence classes for this relation on V if $n > 0$. An orientation on V is then an injective assignment of $+1$ and -1 to the equivalence classes. How does this definition work in the case $n = 0$?

The equivalence class that is assigned $+1$ is called positive, the other one negative. Often people also call the postive equivalence class the orientation. The standard orientation on $\mathbb{R}^n$ is defining the equivalence class of the standard basis to be positive.

 (b) Another definition of an orientation on V is the following. Let $\bigwedge^n V$ be the vectorspace of all n-vectors of V. We define an equivalence relation on $(\bigwedge^n V)\backslash\{0\}$ as follows. For $\alpha, \beta \in (\bigwedge^n V)\backslash\{0\}$ we let $\alpha \sim \beta$ if there exists a $c \in \mathbb{R}$ with $c > 0$ such that $\alpha = c\beta$. Show that this is an equivalence relation and that there are 2 equivalence classes for this relation on V if $n > 0$. An orientation on V is then an injective assignment of $+1$ and -1 to the equivalence classes. How does this definition work in the case $n = 0$?

 (c) Show that the two previous definitions are equivalent.

 (d) Show that we find another equivalent definition of an orientation if we replace $\bigwedge^n V$ by $\bigwedge^n V^*$ in the previous definition.

 (e) Sometimes, if a vector space has some additional structure, it has a natural orientation. Show that the complex structure, i.e. the multiplication with $i \in \mathbb{C}$, on $\mathbb{C}^n \cong \mathbb{R}^{2n}$ defines a natural orientation on $\mathbb{R}^{2n}$, i.e. one independent of (other) choices.

33. Let $X \subset \mathbb{R}^n$ be open and connected. Show that the definition of an orientation in question 32(a) can be extended to define what an orientation is on X such that this definition is equivalent to the extension given in question 26 of the definition in 32(d).

34. Let $X \subset \mathbb{R}^n$ be closed and connected, let X have an orientable interior X° and a piecewise smooth boundary ∂X and let X be such that it equals the closure of its interior.

 (a) Let $\omega \in \Omega_0^n(X^\circ)$ be a representative of the positive orientation of X°. Let v_{out} be a vector field on ∂X, i.e. differentiable on the smooth pieces, such that v_{out} always points outward of X. Show that we can define an orientation on (the pieces of) ∂X by calling $\omega_{\partial X}(x)(v_1 \wedge \cdots \wedge v_{n-1}) := \omega(x)(v_{out} \wedge v_1 \wedge \cdots \wedge v_{n-1})$ positive and show that this orientation is natural, i.e. independent of (other) choices. Note that since ω is differentiable there is a unique extension of ω to the closure of X.

(b) Take on $[0,1]^k$ the orientation induced by $e^1 \wedge \cdots \wedge e^k$. Verify that with this orientation and the induced orientations on the faces $I^k_{i,\sigma}$, the signs in the expression above Definition 17 on page 36 of the lecture notes make sense.

35. Let $F = (F_1, F_2, F_3)$ be a 1-vector field on $\mathbb{R}^3$, and let $f : \mathbb{R}^3 \overset{open}{\supseteq} P \to \mathbb{R}$ be smooth. Define the forms

$$\omega^1_F = F_1 dx + F_2 dy + F_3 dz$$
$$\omega^1_F = F_1 dy \wedge dz + F_2 dz \wedge dx + F_3 dx \wedge dy$$

(a) Prove that

i. $df = \omega^1_{\text{grad } f}$;

ii. $d\omega^1_F = \omega^2_{\text{curl } F}$;

iii. $d\omega^2_F = \text{div}(F) dx \wedge dy \wedge dz$.

(b) Prove that curl grad$f = 0$ and that div curl $F = 0$.

(c) Assume that $P \subset \mathbb{R}^3$ is open and star shaped, and let $F = (F_1, F_2, F_3)$ be a 1-vector field on P. Show that there exists a smooth $f : P \to \mathbb{R}^3$ such that $F = \text{grad } f$ if curl $F = 0$. Similarly, show that there exists a 1-vectorfield G on P such that $F = \text{curl } G$ if div $G = 0$.

36. Let $U \subset \mathbb{R}^n$ be open, and let $f : U \to \mathbb{R}^n$ be a diffeomorphism onto its image. Prove that every closed form on U is exact if and only if every closed form on $f(U)$ is exact.

37. Let $k \in \mathbb{N}$ and ω_1, ω_2 be k-forms. Prove that $d\omega_1 = d\omega_2$ if and only if $\omega_1 - \omega_2$ is exact.

38. (Brouwer's fixed point theorem).

- Let $n \in \mathbb{Z}_{\geq 1}$ and $D^n \subset \mathbb{R}^n$ be the n-dimensional unit disk. Prove that there exists no smooth function $f : D^n \to \partial D^n$ such that $f|_{\partial D^n} = \text{id}|_{\partial D^n}$. (Hint: find a $n-1$-form on ∂D^n that is nowhere equal to zero. (Hint of hint: look at Exercise 34.) Then consider $\int_{D^n} d(f^*\omega)$.)
- Prove Brouwer's fixed point theorem: for all $n \in \mathbb{N}$, every smooth function $f : D^n \to D^n$ has a fixed point.