Exam introduction to differentiable manifolds 1, 1-15-2019

Question 1

For fixed real number $q > 0$ consider the following system of equations:

$$\begin{aligned} q^x + q^y + q^z &= 2q + q^q \\ -x^q + y^q + z^q &= q^q \end{aligned}$$

a. Prove the following statement about the set S of solutions to the above equations: When $q \neq 1$, there is an open neighborhood $U \subset \mathbb{R}^3$ of the point $(1, 1, q) \in S$ such that $U \cap S$ is C^1 diffeomorphic to an open interval of $\mathbb{R}$.

b. Formulate and prove a similar statement in the case $q = 1$.

Part a. If we define the function $\mathbb{R}^3 \ni (x, y, z) \overset{F}{\longmapsto} (q^x + q^y + q^z, -x^q + y^q + z^q) \in \mathbb{R}^2$ then $S = F^{-1}(\{2q + q^q, q^q\})$. This function is C^1 differentiable and with respect to the standard bases its derivative $F'(x, y, z)$ has matrix given by the partial derivatives of its components:

$$\begin{pmatrix} x^q \log q & y^q \log q & z^q \log q \\ -qx^{q-1} & qy^{q-1} & qz^{q-1} \end{pmatrix}$$

Setting $(x, y, z) = (1, 1, q) = p$ in this matrix we see that $\ker F'(p)$ has dimension 1 when $q \neq 1$ because the first two columns are independent. The third column is q^q times the second. More precisely the kernel is spanned by the vector $w = q^q e_2 - e_3$. Extending w to a basis of $\mathbb{R}^3$ yields an isomorphism $\alpha : \mathbb{R}^3 \to \mathbb{R}^3$ sending the kernel to the first component $\mathbb{R} \times \{0\}$. The implicit function theorem then states that there exists an open interval $0 \in X \subset \mathbb{R}$ and an open set Y and a C^1 function $f : X \to Y$ such that $\alpha(S) \cap (X \times Y + \alpha(p)) = \alpha(p) + \{(x, f(x)) | x \in X\}$. The required C^1 diffeomorphism is then $\phi : X \to S$ defined by $\phi(t) = \alpha^{-1}(\alpha(p) + (t, f(t)))$. This ϕ is a bijection because the inverse is given by $\phi^{-1}(x, y, z) = e^1(\alpha(x, y, z) - \alpha(p))$. Both are C^1 because they are a composition of C^1 functions.

Part b. In case $q = 1$ the first equation becomes the trivial equation $3 = 3$ so there is only one equation to consider. The linear equation $-x + y + z = 1$. The solution set is the plane orthogonal to $(-1, 1, 1)$ passing through the points $-e_1, e_2, e_3$. It may also be viewed as a translate of the plane $\ker F_2'$. Choosing a basis for the two-dimensional linear subspace gives a C^1 diffeomorphism with $\mathbb{R}^2$ and translation is also a smooth diffeomorphism.

Question 2

a. Explain how the formula ydx can be interpreted as a C^1, 1-covector field ω on $\mathbb{R}^2$.

b. Express $d\omega$ as a wedge product of two 1-covector fields.

c. Suppose $\gamma : [0,1] \to \mathbb{R}^2$ is the 1-cube defined by $\gamma(t) = (t, -(t-1)t)$. Calculate the integral $\int_\gamma \omega$ directly from the definition.

d. Calculate the integral $\int_\gamma \omega$ using Stokes theorem for 2-chains.

Part a. First we view x as the function $e^1 : \mathbb{R}^2 \to \mathbb{R}$ and take its differential to get $dx = de^1$, which is a 1-covector field. It is defined by $de^1(p)(v) = (e^1)'(p)(v) = e^1(v)$ since the derivative of a linear function is linear. So $de^1 = e^1$. Next the formula ydx means the 1-covector field ω defined in point $p = (x,y)$ by $\omega(p) = e^2(p)e^1 = ydx$. It is C^1 because its components with respect to the standard basis are.
Part b. $d\omega = d(ydx) = dy \wedge dx$ by definition of the exterior derivative.
Part c. By definition $\int_\gamma \omega = \int_{[0,1]} \mathcal{I}(\gamma', \omega(\gamma))$. The derivative is $\gamma'(t) = e_1 + (-2t+1)e_2$ and so for fixed t the linear map $f_t = \omega(\gamma(t)) \circ \gamma'(t) : \mathbb{R} \to \mathbb{R}$ is given by $f_t = -(t-1)te^1$. The intersection is its determinant which is $-(t-1)t$. So the integral is $\int_{[0,1]} -(t-1)t = -\frac{1}{3} + \frac{1}{2} = \frac{1}{6}$.
Part d. The 2-cube η defined by $\eta(s,t) = (s, -s(s-1)t)$ has boundary $\partial\eta = -\eta_{1,0} - \eta_{2,1} + \eta_{1,1} + \eta_{2,0}$ where $\eta_{1,0}$ and $\eta_{1,1}$ are both constant and $\eta_{2,1} = \gamma$. First $\int_{\eta_{2,0}} \omega = 0$ since $\omega(x,0) = 0$ so using Stokes theorem we may write:

$$\int_\eta d\omega = \int_{\partial\eta} \omega = -\int_\gamma \omega$$

The integral over the 2-chain may be computed explicitly by computing the matrix of $\eta'(s,t)$, which is $\begin{pmatrix} 1 & 0 \\ -(2s-1)t & -s(s-1) \end{pmatrix}$. Now $\mathcal{I}(\eta', \omega(\eta)) = \mathcal{I}(e_1 \wedge (-(2s-1)te_1 - s(s-1)e_2, e^2 \wedge e^1) = s(s-1)$ so $\int_\eta d\omega = \int_{[0,1]^2} s(s-1) = -\frac{1}{6}$ by Fubini.

Question 3

a. Consider a metric g on open set $P \subset \mathbb{R}^n$ and an isometry $\phi : P \to P$. If γ is a 1-cube of minimal length between points $p, q \in P$, show that $\phi \circ \gamma$ is also a differentiable curve of minimal length between $\phi(p), \phi(q)$.

b. Find an element of $Z \in \Lambda^2(\mathbb{R}^4)$ such that $Z \wedge Z \neq 0$ and $\star Z = Z$ with respect to the standard orientation and Euclidean metric.

Part a. ϕ is an isometry so for any point x we have $g(x)(v,w) = g(\phi(x))(\phi'(x)v, \phi'(x)w)$. The length of a curve α is given as $L(\alpha) = \int_{[0,1]} g(\alpha)(\alpha', \alpha')^{\frac{1}{2}} = \int_{[0,1]} g(\phi\circ\alpha)(\phi' \circ \alpha', \phi' \circ \alpha')^{\frac{1}{2}} = L(\phi \circ \alpha)$ using the chain rule. For a contradiction, suppose there is a curve α between $\phi(p), \phi(q)$ with $L(\alpha) < L(\phi \circ \gamma)$. Then applying the isometry ϕ^{-1} we also find a curve $\phi^{-1} \circ \alpha$ from p to q whose length is less than that of γ.
Part b. Take $Z = \frac{1}{\sqrt{2}}(e_1 \wedge e_2 + e_3 \wedge e_4)$. Then $Z \wedge Z = e_1 \wedge e_2 \wedge e_3 \wedge e_4$ and $\star e_1 \wedge e_2 = e_3 \wedge e_4$ and $\star e_3 \wedge e_4 = e_1 \wedge e_2$ so $\star Z = Z$.

Question 4

Consider the manifold M with atlas defined by charts $M^1 = (0,1)$ and $M^2 = (1,2)$ and $M^1_2 = (0,1) - \{\frac{1}{2}\}$ and $M^2_1 = (1,2) - \{\frac{3}{2}\}$ and transition map $\tau^1_2 : M^1_2 \to M^2_1$ given by $\tau^1_2(t) = \begin{cases} t + \frac{3}{2} \text{ if } t < \frac{1}{2} \\ t + \frac{1}{2} \text{ if } t > \frac{1}{2} \end{cases}$.

a. What is the dimension of M? and is M a C^2 manifold?

b. Give an atlas for the tangent bundle TM of M.

c. Write down an explicit example of a C^1 differentiable 2-covector field ω on TM that is not everywhere zero.

d. Can you find a function $f : M \to TM$ and find an ω as in part c. so that $f^*\omega$ is not everywhere zero?

Part a. The dimension of M is one since each of the charts in the atlas is an open subset of $\mathbb{R}^1$. The transition function is a translation which is C^2 as the first derivative is the identity.

Part b. The tangent bundle $B = TM$ has by definition an atlas whose charts are $B^i = M^i \times \mathbb{R}$ and $B^i_j = M^i_j \times \mathbb{R}$. The transition map is $\epsilon^1_2 : B^1_2 \to B^2_1$ and is given by $\epsilon^1_2(p, v) = (\tau^1_2(p), (\tau^1_2)'(p)v)$. The fact that this is an atlas follows from the chain rule.

Part c. Define ω locally by setting $\omega^i : B^i \to \Lambda^2(\mathbb{R}^2)^*$ to be $\omega^i(z) = e^1 \wedge e^2$ for any $z \in B^i$. Here $i = 1, 2$. The two local descriptions of the 2-covector field are consistent since the derivative of τ^1_2 is the identity.

Part d. No you cannot because the pull-back $f^*\omega$ will be a 2-covector field on a 1-dimensional manifold and these are always 0.