The Drinfeld double construction according to knot theory

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Abstract

We give a knot theoretical motivation for the Drinfeld double construction of quasi-triangular Hopf algebras.

1 Introduction

In the theory of Hopf algebras Drinfeld's double construction is of fundamental importance. It allows one to produce a solution to the Yang-Baxter equation starting from only a single Hopf algebra. Solutions to the Yang-Baxter equation allow one to find representations of the braid group and therefore Drinfeld's double construction is highly relevant for knot theory. In fact all of the standard quantum groups that are at the basis of the quantum knot invariants arise this way.

In this note we rephrase the double construction in purely knot theoretical terms. This is done by interpreting all Hopf algebra operations in terms of operations on tangles. The key idea is to start with a special class of tangle diagrams is called over-then-under (OU) tangles. These are tangles that have a diagram that does not include the configuration shown in Figure 1. In words this means that, all the overpasses come first as one walks along a strand of the tangle. In some sense OU the opposite of an alternating tangle.

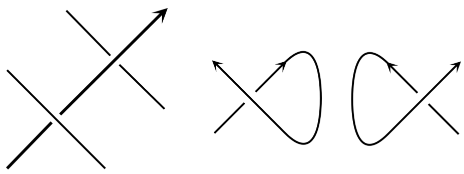


Figure 1: The three forbidden tangles. A tangle diagram is OU if it does not contain any of these three.

Some elementary (non)-examples of OU-diagrams are shown in Figure 2.

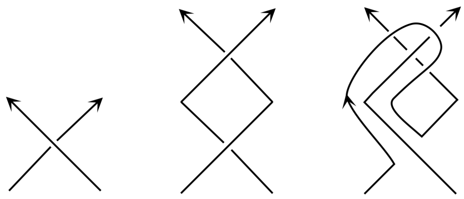


Figure 2: Left: A single (negative) crossing seen as a 2-strand OU tangle diagram. Middle: A tangle diagram that is *not* OU. Right: An isotopic tangle that is OU.

OU tangles are remarkably rigid in the sense that any Reidemeister III move destroys the property and if it exists, a minimal crossing number OU diagram is unique. An idea for bringing any tangle diagram into OU-form by isotopies is as follows. The Reidemeister 1 kinks can be closed in the other direction and more interestingly the first forbidden tangles can be removed using the glide move, which is the isotopy shown in Figure 3. Unfortunately such an algorithm may not terminate. Nevertheless the glide move will be the key to our knot theoretical approach to the Drinfeld double. For more on OU tangles see Cite OU paper.

In this paper we start by constructing an invariant of OU tangles from a Hopf algebra. The algebraic equivalent of the glide move will then allow us to extend our invariant to all tangles. The result is the universal invariant of the Drinfeld double of the Hopf algebra we started with. Moreover the Hopf operations take on a natural meaning in terms of tangle operations. Applying coproduct, counit and antipode to our invariant gives the invariant of the tangle after doubling, deleting and reversing a strand respectively.

We first carry out the above program in a special case in which one of the algebras involved is involutory $S^2 = \text{id}$. To extend our ideas to all Drinfeld doubles we make use of rotational diagrams where we keep track of the (planar) rotation of the arcs in our diagram. This seems important to reflect the algebra of the antipode which is generally not an involution so can not correspond precisely to reversal of a strand. More

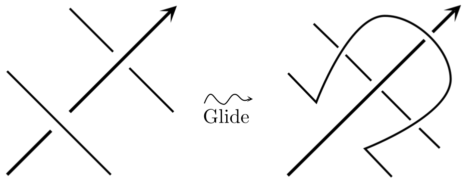


Figure 3: The glide move

concretely it is the non-braid-like Reidemeister II move that fails under the naive set up and forces us to pass to a more subtle framework.

Ordinary tangles will be shown to inject into equivalence classes of rotational diagrams. Therefore the resulting invariant of rotational diagrams is the desired invariant of tangles that we sought for. It has all the desired properties that we met in the special case and is a knot theoretical incarnation of the Hopf algebra of the Drinfeld double. As a bonus the notion of ribbon element and Drinfeld element is also clarified.

2 The involutory case

Throughout this section we consider oriented, framed tangles without closed components. This includes (long)knots as one-strand tangles. We will order the strands and consider our tangles up to isotopy and reordering of the strands.

We start with a finite dimensional Hopf algebra[1] $\mathbb{U}$ over a field κ and consider its dual $\mathbb{O} = \mathbb{U}^*$. The dual pairing is denoted $\langle .,. \rangle : \mathbb{U} \times \mathbb{O} \to \kappa$ and we will often work with a basis $u^1, \dots u^n$ of $\mathbb{U}$ together with the dual basis $o^1 \dots, o^n$ of $\mathbb{O}$. Throughout this section $\mathbb{U}$ will be assumed involutory $S^2 = \mathrm{id}$, later we will drop this assumption.

As a simple running example we choose $\mathbb{O} = \kappa(G)$ to be the κ valued functions on some finite group G and $\mathbb{U} = \kappa[G]$ is the group algebra. The dual pairing in this case is given by evaluation of the functions on the elements of the group. Denoting by δ_g the function that takes 1 on g and is 0 we may write the coproduct as $\Delta_{\mathbb{U}}(g) = g \otimes g$ and $\Delta_{\mathbb{O}}(\delta_g) = \sum_{ab=g} \delta_a \delta_b$. The antipode is just inversion and hence an involution $S_{\mathbb{U}}(g) = g^{-1}$ and $S_{\mathbb{O}}(\delta^g) = \delta^{g^{-1}}$ and finally $\varepsilon_{\mathbb{U}}(g) = 1$ and $\varepsilon_{\mathbb{O}}(\delta^g) = \delta^g(e)$.

Consider the elements $X, \bar{X} \in \mathbb{U} \otimes \mathbb{O}$ defined by $X = \sum_i o^i \otimes u^i$ and $\bar{X} = \sum_i o^i \otimes S_{\mathbb{U}}(u^i)$. Notice X is just the canonical 'identity' element dual to the pairing and $\bar{X}$ is its multiplicative inverse. This is because $\sum_{i,j} o^i o^j \otimes u^i \otimes o^j = \sum_k o^k \otimes \Delta_{\mathbb{U}}(u^k)$ and applying $(\mathrm{id} \otimes m) \circ (\mathrm{id} \otimes \mathrm{id} \otimes S)$ to the left hand side gives $X\bar{X}$ while the antipode axiom implies the right right hand side becomes $1 \otimes 1$. In our running example we have $X = \sum_{g \in G} \delta_g \otimes g$ and $\bar{X} = \sum_{g \in G} \delta_g \otimes g^{-1}$.

At first we will define the double of $\mathbb{U}$ as a vector space $\mathbb{D}$ spanned by finite formal sums of products ou with $o \in \mathbb{O}$ and $u \in \mathbb{U}$. Both $\mathbb{O}$ and $\mathbb{U}$ are included in $\mathbb{D}$, making $\mathbb{D}$ both a left $\mathbb{O}$ module and a right $\mathbb{U}$ module.

With the above algebraic data we can assign a value $Z_{\mathbb{D}}$ to each OU-tangle diagram as follows.

Definition 1. *For an OU-tangle diagram T with n strands define $Z_{\mathbb{D}} \in \mathbb{D}^{\otimes n}$ where n is the number of strands of T. $Z_{\mathbb{D}}$ computed in three steps:*

1. *Place a pair (o^i, u^i) on every positive crossing of T and a pair $(o^i, S(u^i))$ on every negative crossing, associating for each pair the element in $\mathbb{O}$ with the overpassing strand near the crossing and the element in $\mathbb{U}$ with the under-pass.*
2. *For each strand multiply the elements associated to it in order of appearance to obtain an element in $\mathbb{D}$. Now take the tensor product of these elements of $\mathbb{D}$ in the order of the strands.*
3. *$Z_{\mathbb{D}}$ is the sum of the contributions in part 2) where we sum over all ways of placing pairs according to the rules in part 1).*

Perhaps the simplest example of this construction is $Z_{\mathbb{D}}(T) = X$ when T is a single positive crossing. A more interesting example is provided by the rightmost tangle T in Figure 2. In Figure 4 we have redrawn it in a more convenient form indicating the elements used in computing $Z_{\mathbb{D}}$. There are four crossings in the tangle T, three positive and one negative so we place pairs (o^i, u^i) on the positive crossings shown in red blue and green. On the negative crossing we place a pair $(o^d, S(u^d))$ as shown in brown. Next we multiply the elements found on the first strand and get $o^c o^b o^d S(u^d) u^a u^c \in \mathbb{D}$. Doing the same on the second strand gives $o^a u^c \in \mathbb{D}$. Finally we take the tensor product of these and sum over the indices of the four pairs a, b, c, d to find

$$Z_{\mathbb{D}}(T') = \sum_{a,b,c,d} o^c o^b o^d S(u^d) u^a u^c \otimes o^a u^c \in \mathbb{D}^{\otimes 2}$$

[1]Recall this means both $\mathbb{U}$ and its dual are associative algebras with unit. The duals of the product and the unit are called coproduct Δ and counit ε, they are assumed algebra morphisms. Finally there exists a map called antipode $S : \mathbb{U} \to \mathbb{U}$ satisfying $m(S \otimes \mathrm{id})\Delta = 1\varepsilon = m(\mathrm{id} \otimes S)\Delta$.

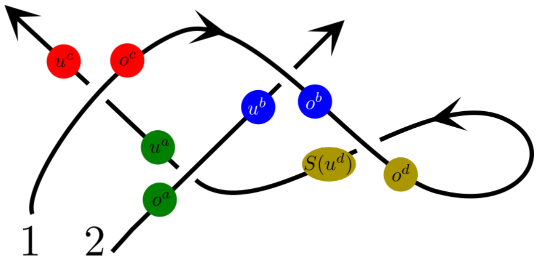


Figure 4: Computing the invariant $Z_{\mathbb{D}}$ by placing copies of X, $\bar{X}$ on the crossings and multiplying along the strands.

The expressions for $Z_{\mathbb{D}}$ look like state sums but another way to think of them is that $Z_{\mathbb{D}}$ of a diagram is assembled from the crossings by multiplying copies of the elements X and $\bar{X}$. It is understood that the empty product corresponds to the unit element so that is the value $Z_{\mathbb{D}}$ assigns to a single crossingless strand. Our definition is merely a version of the universal invariant corresponding to the algebra $\mathbb{D}$ with universal R-matrix X [3].

$Z_{\mathbb{D}}$ is invariant under both oriented Reidemeister moves, IIb and IIc shown in Figure 5. This notation is taken from [4] where these moves are shown to be sufficient. For IIb this is just $\bar{X}X = 1 \otimes 1$ as was already asserted above. Applying id $\otimes S_{\mathbb{U}}$ to both sides of this equation and making use of the fact that $S_{\mathbb{U}}$ is an involution and $S(ab) = S(b)S(a)$ also proves IIc. In the general case where $S_{\mathbb{U}}$ is not an involution IIc is not satisfied and this forces us to consider more subtle types of tangle diagrams, see section 3.

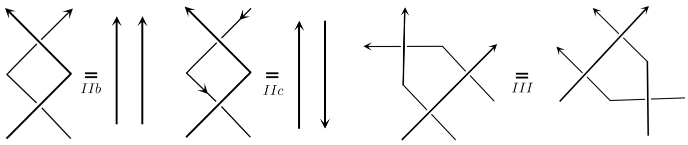


Figure 5: The oriented Reidemeister moves IIb and IIc and III.

So far we know that $Z_{\mathbb{D}}$ is an invariant of (framed) OU-tangles. Recall that the glide move shown in Figure 3 can be used to (almost) turn general diagrams into OU diagrams. The algebraic counterpart of the glide move will upgrade $Z_{\mathbb{D}}$ to an invariant of all tangles in the sense that it will be well-defined and under all oriented Reidemeister moves (ignoring Reidemeister I of course).

The glide move is an isotopy between a non-OU tangle and an OU tangle so if we are to extend $Z_{\mathbb{D}}$ to all tangles it should take the same value on both sides of the glide move. In Figure 6 we placed pairs of algebra elements on the crossings to easily compute $Z_{\mathbb{D}}$ of both sides.

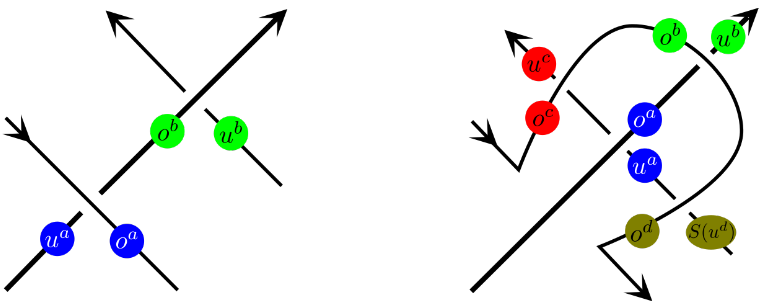


Figure 6: Representing the glide move in algebra.

Of course the left hand side is not an OU-tangle so $Z_{\mathbb{D}}$ is not yet defined but we ignore this for a moment and simply place the pairs on the crossings as we would normally do it. Multiplying as usual the left hand side gives

$$\sum_{a,b} u^a o^b \otimes o^a \otimes u^b$$

The final two factors are in $\mathbb{D}$ and the first tensor factor contains precisely what we do not know yet: the product $u^a o^b$. The right hand side is OU and gives:

$$\sum_{a,b,c,d} o^a u^b \otimes o^c o^b o^d \otimes S(u^d) u^a u^c \in \mathbb{D}^{\otimes 3}$$

Pairing with $u \in \mathbb{U}$ on the second tensor factor and with $o \in \mathbb{O}$ on the third to turn the left hand side into

ou (using $o = \sum_i \langle o, u^i \rangle u^i$). The right hand side turns into

$$\sum_{a,b,c,d} o^a u^b \langle o^c o^b o^d, u \rangle \langle o, \bar{u}^d u^a u^c \rangle$$

Comparing both sides leads us to the following:

Definition 2. *Define an algebra structure on $\mathbb{D}$ by requiring $\mathbb{O}, \mathbb{U}$ are subalgebras by the inclusions $o \mapsto o1$ and $u \mapsto 1u$ and*

$$uo = \sum_{a,b,c,d} o^a u^b \langle o^c o^b o^d, u \rangle \langle o, S(u^d) u^a u^c \rangle \tag{1}$$

Notice $Z_{\mathbb{D}}$ now extends to an invariant of framed tangles once we prove that the product in the algebra is *associative* because both sides of Reidemeister III are related by a single glide move and a IIc move, see Figure 7. In that case $Z_{\mathbb{D}}$ coincides with what is known as the universal knot invariant corresponding to the algebra $\mathbb{D}$ with R-matrix X[2].

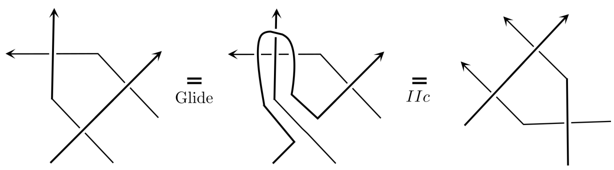


Figure 7: Reidemeister III is implied by a glide and a IIc move.

In our running example this defines the multiplication on $\mathbb{D}$ on the basis to be determined by

$$h\delta_g = \sum_{a,b,c,d \in G} \delta_a b(\delta_c \delta_b \delta_d)(h) \delta_g(d^{-1}ac) = \delta_{hgh^{-1}} h$$

using $c = b = d = h$ and $d^{-1}ac = g$. We invite the reader to use the Wirtinger presentation of the knot group to check that $Z_{\mathbb{D}}(K) = \sum_{\phi:\pi_1(K) \to G} \delta_{\rho(m)} \rho(\ell)$, where the sum is over all group homomorphisms ρ.

2.1 Topological interpretation of the Hopf algebra structure

Before going deeper into the algebra let us clarify the nature of the Hopf algebra operations on a class of tangles that is even simpler than OU-tangles. This allows us to use $Z_{\mathbb{D}}$ as an elegant graphical calculus for doing manipulations with Hopf algebra expressions. Not the usual graphical calculus where the coproduct is a Y-shape and the multiplication is an upside down Y-shape but rather the O/U-tangles themselves. Using $Z_{\mathbb{D}}$ we will find a simple topological interpretation for all the fundamental Hopf algebra maps: multiplication is merging of strands, coproduct doubles a strand, counit deletes a strand and antipode reverses it.

Definition 3. *Define an O/U-tangle to a tangle in which every strand consists of either only over-passes or only under-passes or no crossings at all.*

This notion is more restrictive than OU-tangles. A single crossing is a simple example of an O/U tangle. Other examples are obtained by merging two crossings at the over-strands.

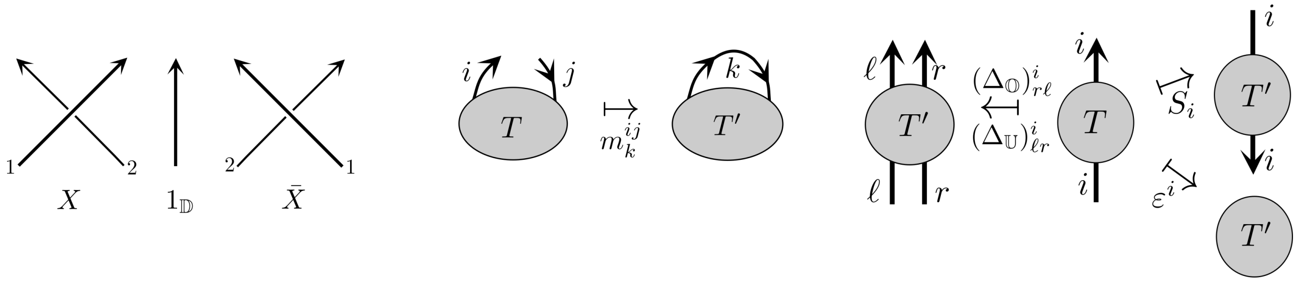


Figure 8: Graphical interpretations for the operations in the tensor algebra of our Hopf algebra.

The graphical calculus of $Z_{\mathbb{D}}$ on O/U-tangles is laid out in the following Lemma, and Figure 8:

Lemma 1. *In what follows T, T' refer to O/U diagrams. Also a Hopf algebra operation applied to a tensor factor corresponding to an under-strand will be assumed to be that of $\mathbb{U}$ and in case of an overstrand it will be that of $\mathbb{O}$.*

1. *$Z_{\mathbb{D}}$ of a crossing is X if the crossing is positive and $\bar{X}$ otherwise.*
2. *$Z_{\mathbb{D}}$ of a crossingless strand is $1_{\mathbb{O}} 1_{\mathbb{U}} \in \mathbb{D}$.*

[2]We prefer the letter X instead of the traditional R for the universal R-matrix.

3. $Z_{\mathbb{D}}(T \sqcup T') = Z_{\mathbb{D}}(T) \otimes Z_{\mathbb{D}}(T')$

4. *If T' is obtained from T by...*

   (a) **connecting**[3] *the end of strand i with the start of j calling the resulting strand k. Then $Z_{\mathbb{D}}(T') = m_k^{ij}(Z_{\mathbb{D}}(T))$ where m_k^{ij} means we should multiply the elements in tensor factor i with those in tensor factor j and place the result in tensor factor k.*

   (b) **deleting** *strand i. Then $Z_{\mathbb{D}}(T') = \varepsilon^i(Z_{\mathbb{D}}(T))$, where ε^i means applying the co-unit to the tensor factor corresponding to strand i.*

   (c) **reversing** *strand i. Then $Z_{\mathbb{D}}(T') = S_i(Z_{\mathbb{D}}(T))$, where S_i means applying the S to the tensor factor corresponding to strand i.*

   (d) **doubling** *strand i, calling the resulting strands ℓ and r where ℓ is to the left of r as seen from the point of view of the framed strand i. Then $Z_{\mathbb{D}}(T') = (\Delta_{\mathbb{U}})^i_{\ell r} Z_{\mathbb{D}}(T)$ if i was an under-strand and $Z_{\mathbb{D}}(T') = (\Delta^{op}_{\mathbb{O}})^i_{\ell r}(Z_{\mathbb{D}}(T))$ if i was an over-strand. Here $\Delta^i_{\ell r}$ means applying Δ to the tensor factor corresponding to strand i and placing the first factor of the result in tensor factor ℓ and the second factor of the result in place r. Also $\Delta^{op}_{\mathbb{O}}$ is defined by $\langle \Delta^{op}_{\mathbb{O}}(o), u \otimes v\rangle = \langle o, vu\rangle$.*

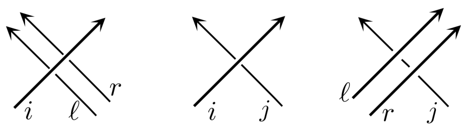
Proof. Items 1,2,3,4a follow directly from the definition of $Z_{\mathbb{D}}$ on O/U-tangle diagrams given in Definition 1. The remaining items 4b,c,d follow from the fact that ε, S, Δ are algebra (anti)-morphisms once we checked the statement is true for the case where T is a single crossing or a crossingless strand. Starting with the co-unit, for T the positive crossing the statement follows from $\sum_i o^i \otimes \varepsilon(u^i) = \sum_i o^i \langle 1_{\mathbb{O}}, u^i\rangle = 1_{\mathbb{O}}$. Since $\varepsilon(S(u)) = u$ the same goes for the negative crossing and the same also holds for applying $\varepsilon_{\mathbb{O}}$ to overstrands. For crossingless strands we just need $\varepsilon(1) = 1 \in \kappa$. Next 4c: The definition of $\bar{X}$ and $S^2 = \mathrm{id}$ and $S(1) = 1$ prove the statement for crossings and crossingless strands. Finally 4d follows from $\Delta(1) = 1 \otimes 1$ and for $T = X$ and the under-strand

$$(\mathrm{id} \otimes \Delta_{\mathbb{U}})Z_{\mathbb{D}}(T) = \sum_a o^a \otimes \Delta_{\mathbb{U}}(u^a) = \sum_{a,b,c} o^a \otimes \langle o^b o^c, u^a\rangle u^b \otimes u^c = \sum_{b,c} o^b o^c \otimes u^b \otimes u^c = Z_{\mathbb{D}}(T')$$

Applying $\Delta_{\mathbb{O}}$ to the over-strand we notice the tensor factors need to be swapped to actually coincide with $Z_{\mathbb{D}}(T')$ so we apply the opposite coproduct instead:

$$(\Delta^{op}_{\mathbb{O}} \otimes \mathrm{id})Z_{\mathbb{D}}(T) = \sum_a \Delta^{op}_{\mathbb{O}}(o^a) \otimes u^a = \sum_{a,b,c} \langle o^a, u^c u^b\rangle o^b \otimes o^c \otimes u^a = \sum_{b,c} o^b \otimes o^c \otimes u^c u^b = Z_{\mathbb{D}}(T')$$

The case of the negative crossing is similar since for any Hopf algebra $\Delta(S) = (S \otimes S)\Delta^{op}$. The opposite order in the output of $\Delta_{\mathbb{O}}$ appears in the interpretation because there is an asymmetry in the crossing itself, see Figure 9. Walking along the over-strand after doubling the under-strand one first meets the left-most of the two parallel strands. Conversely, when walking along the under-strand after doubling the over-strand one first meets the right-most of the two parallel strands. □



$(\Delta_{\mathbb{U}})^j_{\ell r} X$ X $(\Delta_{\mathbb{O}})^i_{r\ell} X$

Figure 9: Graphical interpretations for the operations in the tensor algebra of our Hopf algebra.

The above lemma extends to OU-tangle diagrams almost immediately once we define appropriate notions of Δ, ϵ, S on $\mathbb{D}$. The proof relied on checking the statements locally for the crossings and crossingless strands and then extending by algebra (anti)-morphism properties. Diagrammatically this is natural in the sense that multiplication is concatenation of the diagrams. This motivates the following definitions of maps on $\mathbb{D}$.

Definition 4. *Define $1_{\mathbb{D}} = 1_{\mathbb{O}} 1_{\mathbb{U}} \in \mathbb{D}$, the map $\varepsilon_{\mathbb{D}} : \mathbb{D} \to \kappa$ by $\varepsilon_{\mathbb{D}}(ou) = \varepsilon_{\mathbb{O}}(o)\varepsilon_{\mathbb{U}}(u)$ and $S_{\mathbb{D}} : \mathbb{D} \to \mathbb{D}$ by co-product $S_{\mathbb{D}}(ou) = S_{\mathbb{O}}(o)S_{\mathbb{U}}(u)$ and $\Delta_{\mathbb{D}} : \mathbb{D} \to \mathbb{D} \otimes \mathbb{D}$ by $\Delta_{\mathbb{D}}(ou) = \Delta^{op}_{\mathbb{O}}(o)\Delta_{\mathbb{U}}(u)$.*

As a corollary we thus extend the Hopf operations to all OU-tangles:

Lemma 2. *In what follows T, T' refer to OU diagrams.*

1. *$Z_{\mathbb{D}}$ of a crossing is X if the crossing is positive and $\bar{X}$ otherwise.*
2. *$Z_{\mathbb{D}}$ of a crossingless strand is $1_{\mathbb{D}} \in \mathbb{D}$.*
3. $Z_{\mathbb{D}}(T \sqcup T') = Z_{\mathbb{D}}(T) \otimes Z_{\mathbb{D}}(T')$
4. *If T' is obtained from T by...*

   (a) **connecting** *the end of strand i with the start of j calling the resulting strand k. Then $Z_{\mathbb{D}}(T') = (m_{\mathbb{D}})^{ij}_k(Z_{\mathbb{D}}(T))$.*

[3] without introducing new crossings of course

(b) **deleting** *strand i. Then $Z_{\mathbb{D}}(T') = \varepsilon^i_{\mathbb{D}}(Z_{\mathbb{D}}(T))$.*

(c) **reversing** *strand i. Then $Z_{\mathbb{D}}(T') = (S_{\mathbb{D}})_i(Z_{\mathbb{D}}(T))$.*

(d) **doubling** *strand i, calling the resulting strands ℓ and r where ℓ is to the left of r as seen from the point of view of the framed strand i. Then $Z_{\mathbb{D}}(T') = (\Delta_{\mathbb{D}})^i_{\ell r} Z_{\mathbb{D}}(T)$.*

As a word of caution, reversing a strand may turn an OU-tangle diagram T into a non-OU-tangle diagram T' and the above corollary does not apply to such cases as both diagrams are assumed to be OU.

2.2 Drinfeld double construction

Theorem 1. (Drinfeld double construction, involutory case).

1. *$\mathbb{D}$ is a Hopf algebra with respect to the multiplication in (1) and the maps from Definition 4.*
2. *Moreover dropping the requirment that the tangles are OU Definition 1 extends $Z_{\mathbb{D}}$ to all tangle diagrams and is invariant under Reidemeister IIb,c and Reidemeister III making it an invariant of tangles.*
3. *Finally all the properties of $Z_{\mathbb{D}}$ from Lemma 2 extend to all tangle diagrams.*

Proof. We extend the definition of $Z_{\mathbb{D}}$ to all tangle diagrams as in Definition 1, dropping the OU-requirement. However since we do not yet know whether the multiplication (1) is associative we need to place brackets on the strands to indicate the intended order of multiplication.

To establish associativity it suffices to show that

$$\forall a,b,c: \quad (u^a o^b)o^c = u^a(o^b o^c) \qquad (u^a u^b)o^c = u^a(u^b o^c)$$

because $\mathbb{O}$ and $\mathbb{U}$ are associative subalgebras. In the proof we will use the interpretation of the multiplication in terms of the glide move explained in the discussion preceding Equation (1). We start with by proving equality $Z_{\mathbb{D}}(T) = Z_{\mathbb{D}}(T')$ where T and T' are as shown in Figure 10. Indeed by definition $Z_{\mathbb{D}}(T) = \sum_{i,j,k}(u^i o^j)o^k \otimes o^i \otimes u^j \otimes o^k$ and $Z_{\mathbb{D}}(T') = \sum_{i,j,k} u^i(o^j o^k) \otimes u^j \otimes o^k$ so pairing with $u^a \otimes o^b \otimes o^c$ on the final three tensor factors yields $(u^a o^b)o^c = u^a(o^b o^c)$ as desired.

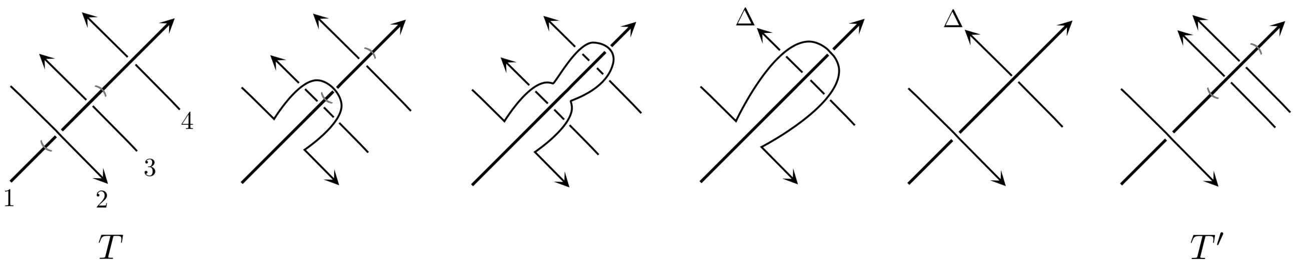


Figure 10: A diagrammatic proof that $Z(T) = Z(T')$. The brackets in grey indicate the order of multiplication/merging of the strands.

Figure 10 does indeed prove that $Z_{\mathbb{D}}(T) = Z_{\mathbb{D}}(T')$ because the value of $Z_{\mathbb{D}}$ on each of the six diagrams is the same. The first equality is by definition of the product: it takes the same value on two diagrams related by a glide move. Going from the second to the third picture we use this property again to perform another glide from $o(uo)$ to $(oo)u$. The fourth diagram should be read as taking $Z_{\mathbb{D}}$ of the diagram shown and then applying $\Delta_{\mathbb{D}}$ to the third tensor factor (corresponding to strand 3). From Lemma 1 we know that $Z_{\mathbb{D}}$ of this coproduct is the same to $Z_{\mathbb{D}}$ of the third diagram because strands 3, 4 are parallel. The fifth diagram once again applies the glide-definition of the product and going to the sixth picture we undo the $\Delta_{\mathbb{D}}$. The proof of $(u^a u^b)o^c = u^a(u^b o^c)$ is analogous and is left to the reader.

We already observed that invariance of $Z_{\mathbb{D}}$ under Reidemeister III is now immediate because both tangles in Reidemeister III are related by a single glide move. Indeed, Reidemeister III is true *by definition* of the multiplication in $\mathbb{D}$.

Now that $\mathbb{D}$ is shown to be associative we can drop the brackets on our tangle diagrams and see that $Z_{\mathbb{D}}$ is well defined on all tangle diagrams.

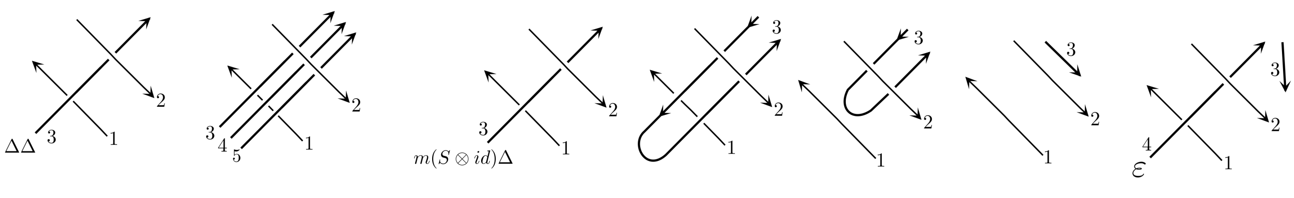


Figure 11: Proofs of the coassociativity and antipode axioms.

Next we check that $\mathbb{D}$ is indeed a Hopf algebra. For this it remains to check coassociativity of Δ_D, the fact that both $\varepsilon_{\mathbb{D}}$ and $\Delta_{\mathbb{D}}$ are algebra morphisms and $m(S_{\mathbb{D}} \otimes \mathrm{id}) \circ \Delta_{\mathbb{D}} = 1_{\mathbb{D}}\varepsilon_{\mathbb{D}}$ and a the same with S on the other side. Coassociativity and the antipode axiom are checked using the OU tangle G shown in Figure 11. In the first diagram we show G itself and using Lemma 2 twice we obtain the second picture regardless of how we apply the two $\Delta_{\mathbb{D}}$ operations to strand 3 as indicated. Pairing on the strands 1 and 2 yields the desired $(\Delta_{\mathbb{D}}\otimes\mathrm{id})\Delta_{\mathbb{D}}(o^a u^b) = (\mathrm{id}\otimes\Delta_{\mathbb{D}})\Delta_{\mathbb{D}}(o^a u^b)$ for all a, b. For the antipode axiom we proceed in the same way using Lemma 2 to get from the third to the fourth picture. Note that the antipode can actually be interpreted as reversal of the strands at the two participating crossings, is we combine Lemma 2 with the definition $S_{\mathbb{D}}(ou) = S_{\mathbb{D}}(u)S_{\mathbb{D}}(o)$. Two applications of Reidemeister II bring us to a diagram that can be interpreted by Lemma 2 as applying ε to strand 3 of G and then tensoring with a unit $1_{\mathbb{D}}$. As before pairing on tensor factors 1 and 2 yields the desired equality.

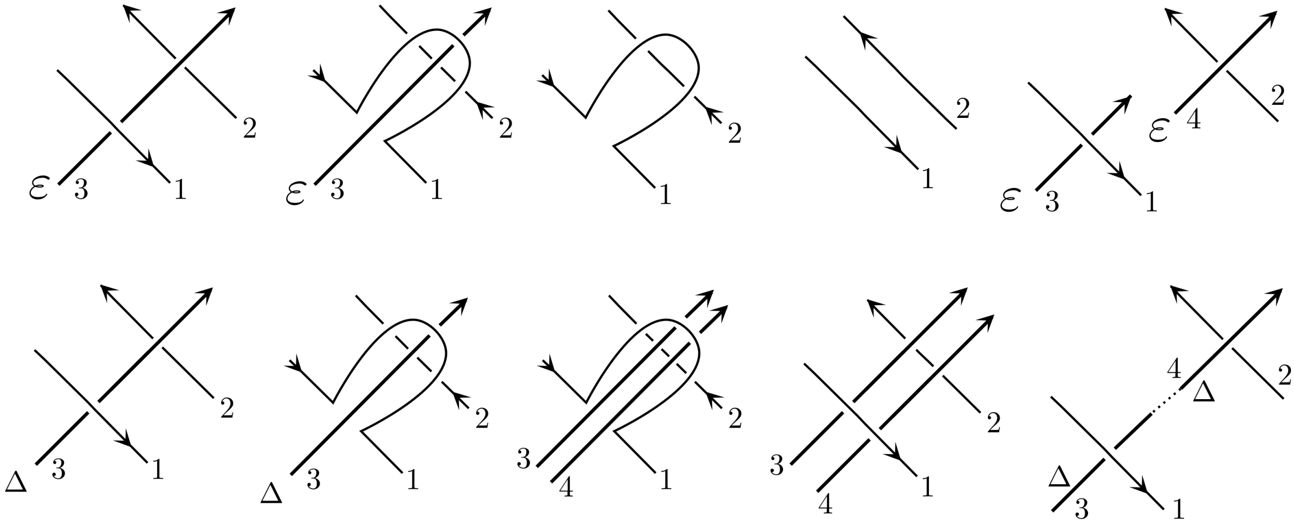


Figure 12: Proving $\varepsilon_{\mathbb{D}}$ and $\Delta_{\mathbb{D}}$ are algebra morphisms.

The proofs of the final two Hopf axioms are contained in Figure 12. This time it suffices to prove $\varepsilon_{\mathbb{D}}(u^a o^b) = \varepsilon_{\mathbb{D}}(u^b)\varepsilon_{\mathbb{D}}(o^a)$ and $\Delta_{\mathbb{D}}(u^a o^b) = \Delta_{\mathbb{D}}(u^a)\Delta_{\mathbb{D}}(o^b)$ for all a, b so to achieve this we work with the tangle H shown in the first column of Figure 12. On the one hand $Z_{\mathbb{D}}(H) = \sum_{ij} o^i \otimes u^j \otimes u^i o^j$ so that applying our morphism $f = \varepsilon$ or Δ will yield $f(u^a o^b)$ after pairing with $u^a \otimes o^b$ on the first two tensor factors. On the other hand H is the left hand side of the glide move so we may turn it into an OU-tangle and then apply Lemma 2 to interpret the morphism f graphically as shown in the third column. The fourth column is obtained by isotopy invariance of $Z_{\mathbb{D}}$ and the fifth column matches this with the desired product $f(u^a)f(o^b)$ after pairing. The dotted line in the bottom right picture is supposed to indicate that we multiply on both factors after applying $\Delta_{\mathbb{D}}$ to the crossings.

Next, points 1,2,3,4a of Lemma 2 (with the OU requirement dropped) follow directly from the definition of the extended $Z_{\mathbb{D}}$. Since $\mathbb{D}$ is a Hopf algebra it follows that $S_{\mathbb{D}}$ is anti-multiplicative and $\Delta_{\mathbb{D}}$ and $\varepsilon_{\mathbb{D}}$ are multiplicative. Therefore the statement of the Lemma extends without any changes. □

Note that $\mathbb{D}$ does indeed conincide with the Drinfeld double found in the literature [1, 2] although there one usually writes the multiplication more explicitly in terms of the coproduct and Sweedler's notation $(\Delta_{\mathbb{O}}\otimes\mathrm{id})\Delta_{\mathbb{O}}(o) = \sum o^{(1)}\otimes o^{(2)}\otimes o^{(3)}$ and $(\Delta_{\mathbb{U}}\otimes\mathrm{id})\Delta_{\mathbb{U}}(u) = \sum u^{(1)}\otimes u^{(2)}\otimes u^{(3)}$. Then we find

$$uo = \sum_{a,b,c,d} o^a u^b \langle o^c\otimes o^b\otimes o^d, (\Delta_{\mathbb{U}}\otimes\mathrm{id})\Delta_{\mathbb{U}}(u)\rangle\langle (S_{\mathbb{O}}\otimes\mathrm{id}\otimes\mathrm{id})(\Delta_{\mathbb{O}}\otimes\mathrm{id})\Delta_{\mathbb{O}}(o), u^d\otimes u^a\otimes u^c\rangle =$$

$$\sum_{a,b,c,d} o^a u^b \langle o^c, u^{(1)}\rangle\langle o^b, u^{(2)}\rangle\langle o^d, u^{(3)}\rangle\langle S_{\mathbb{O}}(o^{(1)}), u^d\rangle\langle o^{(2)}, u^a\rangle\langle o^{(3)}, u^c\rangle =$$

$$\sum o^{(2)}u^{(2)}\langle o^{(3)}, u^{(1)}\rangle\langle S_{\mathbb{O}}(o^{(1)}), u^{(3)}\rangle$$

Also note that our double $\mathbb{D}$ is a quasi-triangular Hopf algebra with respect to the universal R-matrix X in the sense that

$$(\Delta\otimes\mathrm{id})(X) = X_{23}X_{13} \qquad (\Delta\otimes\mathrm{id})(X) = X_{12}X_{13} \qquad \Delta^{op}(x) = \bar{X}\Delta(x)X$$

Here we used the traditional notation where $X_{13} = \sum_i o^i\otimes 1\otimes u^i$ so the subscripts mean the first factor of X sits in place 1 and the second factor sits in place 3. These equations follow directly from the graphical interpretation of the coproduct given in the theorem above. Comparing to the literature this notion of quasi-triangularity is with respect to the opposite multiplication.

3 The Sweedler algebra and the failure of Reidemeister IIc

In this section we introduce Sweedler example and show that in this case we do not even get an invariant of *OU* tangles. The problem is that it fails Reidemeister IIc. As the Sweedler algebra is the simplest of its kind we will also use it as a new running example to illustrate our constructions.

The 4-dimensional Sweedler algebra $\mathbb{SW}$ generated by s, w with relations

$$s^2 = 1 \quad w^2 = 0 \quad ws = -sw$$

it has basis $1, s, w, sw$. The algebra $\mathbb{SW}$ is self dual with respect to the following pairing. Consider a second copy of the Sweedler algebra generated by σ, ω with the same relations. Then the pairing is $\langle , \rangle : \mathbb{SW} \otimes \mathbb{SW} \to \mathbb{Q}$ is summarized by the following matrix

$\langle ., . \rangle$	1	s	w	sw
1	1	1	0	0
σ	1	-1	0	0
ω	0	0	1	1
$\sigma\omega$	0	0	1	-1

Consequently the elements $X, \bar{X}$ are:

$$X_{ij} = (1 + \sigma_i + s_j - \sigma_i s_j)\frac{1 + \omega_i w_j}{2} \qquad \bar{X}_{ij} = \frac{1 - \omega_i w_j}{2}(1 + \sigma_i + s_j - \sigma_i s_j)$$

This is clear once we expand the brackets and write $X_{ab} = \sum_i o_a^i u_b^i$ where we may take $u^1 = 1, u^2 = s, u^3 = w, u^4 = sw$ and $o^1 = \frac{\sigma+1}{2}, o^2 = \frac{\sigma-1}{2}, o^3 = \frac{\sigma+1}{2}\omega, o^4 = \frac{\sigma-1}{2}\omega$. The failure of Reidemeister IIc is now readily confirmed. One finds $X_{12}\bar{X}_{34} /\!/ m_1^{13} m_2^{24} = 1 - (1 - \sigma_1)s_2\omega_1 w_2 \neq 1$.

4 Virtual Rotational tangles

5 The general case

First work with rotation number 0 on every strand so everything is generated by X and S^2? Introduction of C as square root of double kink with rot 2 and writhe 0.

6 Recycling: older material below here

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