KnotTableData

The program

The building blocks

In[53]:=

$$
\begin{aligned}
&\mathrm{Define}\Big[\\
&X_{i,j} = \mathbb{E}_{\{\}\to\{i,j\}}\Big[0,\ x_j\,(y_i - y_j),\\
&\quad \Big(1 + \frac{1}{4\,(-1+T)^2}\big(2 - 4\,T + 2\,T^2 + 2\,x_i\,y_i - 2\,T\,x_i\,y_i - x_j^2\,y_i^2 + 2\,T\,x_j^2\,y_i^2 - T^2\,x_j^2\,y_i^2 - 2\,T\,x_j\,y_j +\\
&\qquad 2\,T^2\,x_j\,y_j + 4\,T\,x_i\,x_j\,y_i\,y_j + x_j^2\,y_j^2 - 2\,T\,x_j^2\,y_j^2 - 3\,T^2\,x_j^2\,y_j^2\big)\,\epsilon + O[\epsilon]^2\Big)\Big],\\
&\overline{X}_{i,j} = \mathbb{E}_{\{\}\to\{i,j\}}\Big[0,\ \frac{x_j\,(-y_i + y_j)}{T},\ \Big(1 + \frac{1}{4\,(-1+T)^2\,T^2}\\
&\quad \big(-(-1+T)^2\,(2\,T^2 + x_j\,y_i\,(-4\,T + 3\,x_j\,y_i)) - 2\,(-1+T)\,x_j\,(T\,(-1+2\,T) + 2\,x_j\,y_i)\,y_j +\\
&\qquad (1+T)\,(-1+3\,T)\,x_j^2\,y_j^2 + 2\,T\,x_i\,y_i\,((-1+T)\,T - 2\,x_j\,((-1+T)\,y_i + y_j))\big)\,\epsilon + O[\epsilon]^2\Big)\Big],\\
&C_i = \mathbb{E}_{\{\}\to\{i\}}\Big[0,\ 0,\ \Big(1 + \epsilon\,\Big(\frac{1}{2} - \frac{x_i\,y_i}{-1+T}\Big)\Big) + O[\epsilon]^2\Big],\\
&\overline{C}_i = \mathbb{E}_{\{\}\to\{i\}}\Big[0,\ 0,\ \Big(1 - \epsilon\,\Big(\frac{1}{2} - \frac{x_i\,y_i}{-1+T}\Big)\Big) + O[\epsilon]^2\Big],\\
&m_{i,j\to k} = \mathbb{E}_{\{i,j\}\to\{k\}}\Big[0,\ y_k\,(\eta_i + \eta_j) - (-1+T)\,\eta_j\,\xi_i + x_k\,(\xi_i + \xi_j),\\
&\quad 1 + \frac{1}{4\,(-1+T)^3}\big(2\,(-1+T)^2\,(-1+3\,T)\,x_k\,\eta_j\,\xi_i\,(2\,y_k - (-1+T)\,\xi_i) +\\
&\qquad (-1+T)^3\,\eta_j\,\xi_i\,(-4\,T + 2\,(1-3\,T)\,y_k\,\eta_j + (1 - 4\,T + 3\,T^2)\,\eta_j\,\xi_i)\big)\,\epsilon + O[\epsilon]^2\Big]\\
&\Big]
\end{aligned}
$$

The knot invariant Rho1

Both Rho1 and Alexander are palindromic $f(T^{-1})$=f(T) so we only list the monomials with non-negative exponents.

Computing Rho1 of the trefoil knot or together with Alexander:

In[54]:=

```
Rho1@Knot[3, 1]
Rho1AndAlex@Knot[3, 1]
Rho1@KnotsUpTo12⟦1⟧
```

Out[54]= T

Out[55]= {T, -1 + T}

Out[56]= T

A small table of the first 30 knots

In[•]:=
```
Flatten@{#, Rho1AndAlex@#} & /@ KnotsUpTo12⟦ ;; 30⟧ // MatrixForm
```

Out[•]//MatrixForm=

Knot[3, 1]	T	$-1+T$
Knot[4, 1]	0	$-3+T$
Knot[5, 1]	$3T+2T^3$	$1-T+T^2$
Knot[5, 2]	$-4+5T$	$-3+2T$
Knot[6, 1]	$4-T$	$-5+2T$
Knot[6, 2]	$4-4T+4T^2-T^3$	$3-3T+T^2$
Knot[6, 3]	0	$5-3T+T^2$
Knot[7, 1]	$6T+5T^3+3T^5$	$-1+T-T^2+T^3$
Knot[7, 2]	$-16+14T$	$-5+3T$
Knot[7, 3]	$12-16T+8T^2-9T^3$	$3-3T+2T^2$
Knot[7, 4]	$32-24T$	$-7+4T$
Knot[7, 5]	$-28+29T-16T^2+9T^3$	$5-4T+2T^2$
Knot[7, 6]	$20-19T+8T^2-T^3$	$7-5T+T^2$
Knot[7, 7]	$8-3T$	$9-5T+T^2$
Knot[8, 1]	$16-5T$	$-7+3T$
Knot[8, 2]	$12-13T+12T^2-10T^3+8T^4-2T^5$	$-3+3T-3T^2+T^3$
Knot[8, 3]	0	$-9+4T$
Knot[8, 4]	$4-6T+8T^2-3T^3$	$5-5T+2T^2$
Knot[8, 5]	$-24+22T-20T^2+13T^3-8T^4+2T^5$	$-5+4T-3T^2+T^3$
Knot[8, 6]	$32-28T+20T^2-5T^3$	$7-6T+2T^2$
Knot[8, 7]	$12-13T+12T^2-10T^3+4T^4-T^5$	$-5+5T-3T^2+T^3$
Knot[8, 8]	$16-12T+4T^2-T^3$	$9-6T+2T^2$
Knot[8, 9]	0	$-7+5T-3T^2+T^3$
Knot[8, 10]	$-4T+2T^2-T^3$	$-7+6T-3T^2+T^3$
Knot[8, 11]	$44-39T+24T^2-5T^3$	$9-7T+2T^2$
Knot[8, 12]	0	$13-7T+T^2$
Knot[8, 13]	$20-14T+4T^2-T^3$	$11-7T+2T^2$
Knot[8, 14]	$68-57T+28T^2-5T^3$	$11-8T+2T^2$
Knot[8, 15]	$-140+120T-64T^2+21T^3$	$11-8T+3T^2$
Knot[8, 16]	$-36+35T-28T^2+17T^3-6T^4+T^5$	$-9+8T-4T^2+T^3$

Up to 10 crossings the pair (Alexander,Rho1) distinguishes all prime knots. However as expected the Kinoshita-Terasaka knot and its mutant the Conway knot yield the same value.

```
Rho1@Knot[11, NonAlternating, 42]
Rho1@Knot[11, NonAlternating, 34]
```

Out[•]= $-2T^2$

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More details on the knot invariant Z (computed up to first order in ϵ)

Checking Reidemeister 1: (it is satisfied up to an overall factor of T or T^{-1})

In[•]:=
$$X_{1,2}\,\overline{C}_3 \mathbin{//} m_{1,3\to1} \mathbin{//} m_{1,2\to1}$$
$$\overline{X}_{1,2}\,C_3 \mathbin{//} m_{1,3\to1} \mathbin{//} m_{1,2\to1}$$
$$\overline{X}_{1,2}\,\overline{C}_3 \mathbin{//} m_{2,3\to2} \mathbin{//} m_{2,1\to1}$$
$$X_{1,2}\,C_3 \mathbin{//} m_{2,3\to2} \mathbin{//} m_{2,1\to1}$$

Out[•]= $\mathbb{E}_{\{\}\to\{1\}}\left[0,\,0,\,1+O[\epsilon]^2\right]$

Out[•]= $\mathbb{E}_{\{\}\to\{1\}}\left[0,\,0,\,1+O[\epsilon]^2\right]$

Out[•]= $\mathbb{E}_{\{\}\to\{1\}}\left[0,\,0,\,T+O[\epsilon]^2\right]$

Out[•]= $\mathbb{E}_{\{\}\to\{1\}}\left[0,\,0,\,\frac{1}{T}+O[\epsilon]^2\right]$

Checking Reidemeister 2:

In[•]:= $X_{1,2}\,\overline{X}_{3,4} \mathbin{//} m_{1,3\text{->}1} \mathbin{//} m_{2,4\to2}$

Out[•]= $\mathbb{E}_{\{\}\to\{1,2\}}\left[0,\,0,\,1+O[\epsilon]^2\right]$

Checking Reidemeister 3:

In[•]:=
$$\left(X_{1,2}\,X_{4,3}\,X_{5,6} \mathbin{//} m_{1,4\to1} \mathbin{//} m_{2,5\to2} \mathbin{//} m_{3,6\to3}\right) \equiv$$
$$\left(X_{2,3}\,X_{1,6}\,X_{4,5} \mathbin{//} m_{1,4\to1} \mathbin{//} m_{2,5\to2} \mathbin{//} m_{3,6\to3}\right)$$

Out[•]= True

Trefoil knot

In[•]:= $\left(X_{5,1}\,X_{2,6}\,X_{7,3}\,C_4 \mathbin{//} m_{1,2\to1} \mathbin{//} m_{1,3\to1} \mathbin{//} m_{1,4\to1} \mathbin{//} m_{1,5\to1} \mathbin{//} m_{1,6\to1} \mathbin{//} m_{1,7\to1}\right)$

Out[•]= $$\mathbb{E}_{\{\}\to\{1\}}\left[0,\,0,\,\frac{1}{T-T^2+T^3}+\frac{\left(1-2T+2T^2-2T^3+T^4\right)\epsilon}{T-3T^2+6T^3-7T^4+6T^5-3T^6+T^7}+O[\epsilon]^2\right]$$

The same can also be computed using the knot table name of the trefoil:

In[57]:=
```
Z@Knot[3, 1]
Collect[#, ϵ, Factor] & /@ Z@Knot[3, 1]
```

Out[57]= $$\mathbb{E}_{\{\}\to\{0\}}\left[0,\,0,\,\frac{T^3}{1-T+T^2}+\frac{\left(-T^3+2T^4-2T^5+2T^6-T^7\right)\epsilon}{1-3T+6T^2-7T^3+6T^4-3T^5+T^6}+O[\epsilon]^2\right]$$

Out[58]= $$\mathbb{E}_{\{\}\to\{0\}}\left[0,\,0,\,\frac{T^3}{1-T+T^2}-\frac{(-1+T)^2\,T^3\,\left(1+T^2\right)\epsilon}{\left(1-T+T^2\right)^3}\right]$$

The interesting bit Rho1 is the numerator of the coefficient of ϵ. The general form of this coefficient is $\frac{-(1-T)^2\,\mathrm{Rho1}}{\mathrm{Alexander}^3}$.